Points in general position in real-analytic submanifolds in $\mathbb{C}^N$ and applications

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1. Introduction

Two pairs (M, p) and (M', p') of germs of real (locally closed) submanifolds $M, M' \subset \mathbb{C}^N$ at distinguished points $p \in M$ and $p' \in M'$ are said to be *biholomorphically equivalent* (or just *equivalent* for short) if there is a biholomorphic map H between open neighborhoods of p and p' in $\mathbb{C}^N$ sending p to p' and mapping a neighborhood of p in M onto a neighborhood of p' in M'. We write $(M, p) \sim (M', p')$ for equivalent pairs and $H: (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ for a map between open neighborhoods of p and p' in $\mathbb{C}^N$ sending p to p'.

It is easy to construct germs of smooth real curves in $\mathbb{C}$ that are not equivalent. In contrast, any two germs of *real-analytic* curves in $\mathbb{C}$ at arbitrary distinguished points are always equivalent, since any real-analytic diffeomorphism between them extends to a biholomorphism between some open neighborhoods in $\mathbb{C}$.

The simplest example of non-equivalent real-analytic submanifolds of the same dimension is given by $(\mathbb{C}, 0)$ and $(\mathbb{R}^2, 0)$ both linearly embedded in $\mathbb{C}^2$ in the standard way. More generally, it is easy to see that two germs at 0 of real linear subspaces of $\mathbb{C}^N$ are equivalent if and only if they can be transformed into each other by a complex linear automorphism of $\mathbb{C}^N$.

In this paper we give a local description of a real-analytic submanifold $M \subset \mathbb{C}^N$ at a "general" point (see Theorem 2.5 below). This description is based on various notions of nondegeneracy and is of interest in its own right. An important application is that at a "general" point $p \in M$, the germ (M, p) is equivalent to another germ (M', p') if and only if (M, p) and (M', p') are "formally" equivalent (see Theorem 6.1 below.) This result is the main theorem in **[BRZ 2000]**. (It should be noted that there exist pairs (M, p) and (M', p') which are "formally" equivalent, but not biholomorphically equivalent; see §4.1.) We also address here the case of real-algebraic submanifolds and their algebraic equivalences, which was not studied in **[BRZ 2000]**. (See Theorem 9.1 below.)

We mention briefly that the study of biholomorphic equivalence of real submanifolds in $\mathbb{C}^N$ goes back to Poincaré **[P 1907]** and E. Cartan **[Ca 1932a]**, **[Ca 1932b]**, and **[Ca 1937]**. In their celebrated work Chern and Moser **[CM 1974]** solved the equivalence problem for germs of Levi nondegenerate real-analytic hypersurfaces in

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$\mathbb{C}^N$ and showed in particular that in this context the notions of formal and biholomorphic equivalence coincide. We will mention more recent work related to Theorem 6.1 later in this article.

2. Structure decomposition results for points in general position

If M is a connected real-analytic submanifold of $\mathbb{C}^N$, we say that a property holds for $p \in M$ in *general position* if it holds for all p outside a proper real-analytic subvariety of M. A number associated to M is said to be a *biholomorphic invariant* if it is preserved by biholomorphic equivalences of germs of M at any point in M.

2.1. Generic and CR submanifolds. A smooth real submanifold $M \subset \mathbb{C}^N$ is called *generic* (or *generating*, in some translations) if $T_pM + JT_pM = T_p\mathbb{C}^N$ for all $p \in M$, where $J: T\mathbb{C}^N \to T\mathbb{C}^N$ denotes the standard complex structure of $\mathbb{C}^N$, and T_pM denotes the (real) tangent space of M at p. More generally, if the space $T_qM + JT_qM$ has constant dimension for q near p, M is said to be CR at p (or p is a CR point of M). If M is CR at every point, it is said to be a CR submanifold. (For CR manifolds, the reader is referred e.g. to the books **[J 1990]**, **[Bo 1991]**, **[Ch 1991]**, **[BER 1999a]**.) If M is real-analytic, the set of all non CR points of M is a nowhere dense proper real-analytic subvariety of M.

Examples 2.1. In $\mathbb{C}$ all nontrivial smooth submanifolds are generic. More generally, the graph of any (smooth) map between open sets in $\mathbb{C}^n \times \mathbb{R}^{N-n}$ $(0 \le n \le N)$ and $i\mathbb{R}^{N-n}$ is generic. For $N \ge 2$, complex submanifolds of $\mathbb{C}^N$ of positive codimension and their real submanifolds are never generic. The submanifold

$$M := \{w = |z|^2\} \subset \mathbb{C}^2,$$

where (z, w) are taken as coordinates in $\mathbb{C}^2$, is generic and CR everywhere except at the origin in $\mathbb{C}^2$.

The role of generic points is illustrated by the following property.

Proposition 2.1. *If $M \subset \mathbb{C}^N$ is a connected real-analytic submanifold, there exists an integer $0 \le r_1 \le N$ such that for $p \in M$ in general position*

$$(M, p) \sim (M_1 \times \{0\}, 0), \quad M_1 \times \{0\} \subset \mathbb{C}^{N-r_1} \times \mathbb{C}^{r_1}, \tag{2.1}$$

where $M_1 \subset \mathbb{C}^{N-r_1}$ is a generic real-analytic submanifold through 0. The number r_1 with this property is unique and is a biholomorphic invariant.

Remark 2.2. The points p for which the conclusion of Proposition 2.1 holds are in fact the CR points of M.

The number r_1 is called the *excess codimension* of M (cf. **[BRZ 2000]**, §2). It is equal to the maximal codimension of a complex submanifold of $\mathbb{C}^N$ containing an open subset of M. For $p \in M$ in general position, M is CR at p and there exists a complex submanifold of $\mathbb{C}^N$ of codimension r_1 that contains a neighborhood of p in M. This complex submanifold is unique in the sense of germs and is called the *intrinsic complexification* of M at p.

2.2. Finite and minimum degeneracy. Finite nondegeneracy is a higher order generalization of Levi nondegeneracy. For a smooth CR submanifold $M \subset \mathbb{C}^N$ we denote by T^cM the real subbundle given by $T^c_pM = \{X \in T_pM : JX \in T_pM\}$. We consider the $(0,1)$ *vector fields* on M, i.e., the sections of the subbundle

$$T^{0,1}M := \{X + iJX : X \in T^cM\} \subset T^cM \otimes \mathbb{C}.$$

Then M is *Levi nondegenerate* at p if for any $(0,1)$ vector field L with $L(p) \neq 0$, there exists a $(0,1)$ vector field L_1 such that

$$[L_1, \overline{L}](p) \notin T^c_pM \otimes \mathbb{C}.$$

This condition is equivalent to the nondegeneracy of the Levi form defined as the (unique) hermitian form

$$\mathcal{L}_p \colon T^{0,1}_pM \times T^{0,1}_pM \to (T_pM/T^c_pM) \otimes \mathbb{C} \tag{2.2}$$

satisfying

$$\mathcal{L}_p(L_1(p), L(p)) = \frac{1}{2i}\pi[L_1, \overline{L}](p) \tag{2.3}$$

for all $(0,1)$ vector fields L, L_1, where $\pi : TM \otimes \mathbb{C} \to (TM/T^cM) \otimes \mathbb{C}$ is the canonical projection.

The more general concept of finite nondegeneracy can be defined in a similar way as follows.

Definition 2.1. A smooth CR submanifold $M \subset \mathbb{C}^N$ is called *finitely nondegenerate* at p if there exists $l \geq 0$ such that for any $(0,1)$ vector field L on M with $L(p) \neq 0$, there are $(0,1)$ vector fields on M, $L_1, \dots, L_k$, $0 \leq k \leq l$, such that

$$[L_1, \dots, [L_k, \overline{L}] \dots](p) \notin T^c_pM \otimes \mathbb{C}. \tag{2.4}$$

If l is minimal with this property, M is called *l-nondegenerate* at p.

Remark 2.3. It follows that M is 1-nondegenerate if and only if it is Levi-nondegenerate. Also M is 0-nondegenerate if and only if it is totally real, i.e., $T^c_pM = \{0\}$. To check Definition 2.1 it suffices to assume that $L, L_1, \dots, L_k$ in (2.4) are all taken from a fixed local basis of $(0,1)$ vector fields on M near p.

The condition of finite nondegeneracy was given in **[BHR 1996]** for hypersurfaces and can be found in **[BER 1999a]** for CR submanifolds of higher codimension. The formulation given in Definition 2.1 is equivalent to that in the reference above (see § 3 or Proposition 1.24 in **[E 2000]**). In **[BRZ 2000]** this notion was extended to arbitrary real-analytic submanifolds. As in the case of the Levi form, higher order tensors can be used to give an alternative definition of finite nondegeneracy (see §3).

Example 2.4. Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface through 0 given by $\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w)$ where ϕ is a real-valued real-analytic function defined near the origin in $\mathbb{C} \times \mathbb{R}$ satisfying $\phi(z, 0, 0) \equiv 0$. Then M is finitely nondegenerate at 0 if and only if at least one of the partial derivatives $\phi_{z,\bar{z}^k}(0)$ $(1 \leq k < \infty)$ does not vanish. If k is the smallest such integer, then M is k-nondegenerate at 0.

The role of finite nondegeneracy is illustrated by the following property.

Proposition 2.2. *If $M \subset \mathbb{C}^N$ is a real-analytic submanifold, then there exists an integer $0 \leq r_2 \leq N$ such that for $p \in M$ in general position*

$$(M, p) \sim (M_2 \times \mathbb{C}^{r_2}, 0), \quad M_2 \times \{0\} \subset \mathbb{C}^{N-r_2} \times \mathbb{C}^{r_2}, \tag{2.5}$$

where $M_2 \subset \mathbb{C}^{N-r_2}$ is a real-analytic CR submanifold finitely nondegenerate at 0. *The number r_2 with this property is unique and is a biholomorphic invariant.*

The number r_2 is called the *degeneracy* of M. It is equal to the maximal dimension of the leaves of a holomorphic foliation in a neighborhood of a point $p \in M$ such that a neighborhood of p in M is saturated (i.e., is a union of leaves) (see **[F 1977]**). The points $p \in M$ for which the conclusion of Proposition 2.2 holds are said to be of *minimum degeneracy* in M (see **[BRZ 2000]**, §2).

Example 2.5. The hypersurface $M \subset \mathbb{C}^3$ given by $\operatorname{Im} w = |z_1 z_2|^2$ in the coordinates (z_1, z_2, w) is of minimum degeneracy 1 outside the plane $H := \{(0, 0)\} \times \mathbb{R}$. For $p \notin H$, (M, p) is equivalent to $(M_1 \times \mathbb{C}, 0)$ with $M_1 := \{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = |z|^2\}$.

2.3. Finite type and CR orbits. A smooth CR submanifold $M \subset \mathbb{C}^N$ is said to be of *finite type* at p (in the sense of Kohn **[K 1972]** and Bloom–Graham **[BG 1977]**) if all $(0, 1)$ vector fields on M and their conjugates, together with all their higher order commutators span the space $T_pM \otimes \mathbb{C}$. For general CR submanifolds, the condition of finite nondegeneracy and that of finite type are independent, i.e., one condition does not necessarily imply the other. (See Examples 2.6 and 2.7.) However, for hypersurfaces finite nondegeneracy at a point implies finite type at that point.

Example 2.6. The hypersurface $M \subset \mathbb{C}^2$ given in Example 2.4 is of finite type at 0 if and only if at least one of the partial derivatives $\phi_{z^k\bar{z}^l}(0)$, $1 \leq k, l < \infty$, does not vanish. Hence, for instance, the hypersurface given by

$$M = \{(z, w) \in \mathbb{C} \times \mathbb{C} : \operatorname{Im} w = |z|^4\}$$

is of finite type at 0, but is not finitely nondegenerate at 0.

Example 2.7. For any Hermitian form $\langle\, ,\rangle : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^2$ the quadric

$$M : = \{(z, w) \in \mathbb{C} \times \mathbb{C}^2 : \operatorname{Im} w = \langle z, z\rangle\}$$

is not of finite type at any point. If, in addition, the vector-valued Hermitian form $\langle z, z\rangle$ does not vanish identically, then M is finitely nondegenerate at every point.

Proposition 2.3. *If $M \subset \mathbb{C}^N$ is a connected real-analytic submanifold, then there exists an integer $0 \le r_3 \le N$ such that for $p \in M$ in general position*

$$(M, p) \sim (M_3, 0), \quad M_3 \subset \mathbb{C}^{N-r_3} \times \mathbb{R}^{r_3}, \tag{2.6}$$

with $(0, u) \in M_3$ for $u \in \mathbb{R}^{r_3}$ sufficiently small, and for such fixed u, $M_3 \cap (\mathbb{C}^{N-r_3} \times \{u\})$ is a CR submanifold of finite type at $(0, u)$. The number r_3 with this property is unique and is a biholomorphic invariant.

The number r_3 is called the *orbit codimension*. The *CR orbit* of a point $p \in M$ is the germ at p of a (real-analytic) submanifold of M through p of smallest possible dimension to which all the (0, 1) vector fields on M are tangent. The number r_3 is equal to the minimal codimension of any CR orbit. It is also equal to the maximal number r such that there exists a holomorphic submersion of a neighborhood of a point $p \in M$ in $\mathbb{C}^N$ onto $\mathbb{C}^r$ sending M into the real part $\mathbb{R}^r$. (See **[BER 1996]** for the discussion in the algebraic case.) The points p for which the conclusion of Proposition 2.2 holds are said to be of *minimum orbit codimension* (see **[BRZ 2000]**, §2). The germ of the submanifold $M_3 \cap (\mathbb{C}^{r_3} \times \{u\})$ at $(0, u)$ is the CR orbit of M_3 at this point.

In contrast to the situation of Proposition 2.2, in general the submanifold M_3 cannot be "flattened", i.e., written as $M_3' \times \mathbb{R}^{r_3}$, with $M_3' \subset \mathbb{C}^{N-r_3}$ a real-analytic submanifold. One obstruction to such a "flattening" is the possible nonequivalence of the CR orbits $M_3 \cap (\mathbb{C}^{N-r_3} \times \{u\})$ for different $u \in \mathbb{R}^{r_3}$. In order to give an example of such a submanifold that cannot be "flattened" at any point, we introduce some preliminaries.

Consider for any real number u the quadric $M_u \subset \mathbb{C}^6$ of codimension 2 defined in the coordinates $(z, w) \in \mathbb{C}^4 \times \mathbb{C}^2$ by $\operatorname{Im} w = h_u(z, z)$, where h_u is the $\mathbb{C}^2$-valued Hermitian form

$$h_u(z, \chi) := \left(z_1\overline{\chi}_1 + z_3\overline{\chi}_3 + z_4\overline{\chi}_4,\ z_2\overline{\chi}_2 + z_3\overline{\chi}_3 + uz_4\overline{\chi}_4\right). \tag{2.7}$$

The following lemma will be used to construct our example.

Lemma 2.4. *For any $u, u' \in \mathbb{R}$ with $u, u' > 1$ the Hermitian forms h_u and $h_{u'}$ are equivalent if and only if $u = u'$. That is, if there exist a 2×2 invertible real matrix B*

and a 3×3 *complex invertible matrix* A *such that*

$$Bh_{u'}(z, \chi) = h_u(Az, A\chi) \quad \textit{for all } z, \chi \in \mathbb{C}^4, \tag{2.8}$$

then $u = u'$.

Proof. Let $\{e_1, \ldots, e_4\}$ be the standard basis of $\mathbb{C}^4$. We claim first that if $\{v_1, \ldots, v_4\}$ is a basis of $\mathbb{C}^4$ for which $h_u(v_j, v_k) = 0$, $j \neq k$, then there is a permutation σ of $\{1, 2, 3, 4\}$ and complex numbers $\lambda_j \neq 0$ such that $v_j = \lambda_j e_{\sigma(j)}$, $j = 1, \ldots, 4$. To prove the claim, let ϕ_u be the positive definite (scalar) Hermitian form given by $\phi_u(z, \chi) = h_u^1(z, \chi) + h_u^2(z, \chi)$, where h_u^1, h_u^2 are the components of h_u. Then the 4×4 diagonal matrix D with diagonal entries $\big(1, 0, 1/2, 1/(1+u)\big)$ satisfies

$$\phi_u(Dz, \chi) = h_u^1(z, \chi) \quad \text{for all } z, \chi \in \mathbb{C}^4. \tag{2.9}$$

Hence $\phi_u(Dv_j, v_k) = 0$, $j \neq k$, and since ϕ_u is positive definite, it follows that $Dv_j = l_j v_j$, for some $l_j \in \mathbb{C}$, $j = 1, \ldots, 4$. In particular, since D is diagonal and its eigenvalues are distinct, any eigenvector of D is a nonzero multiple of one of the e_j. Since the v_j are all eigenvectors of D, the claim is proved.

Assume that (2.8) holds. Since $h_{u'}(e_j, e_k) = 0$, $j \neq k$, we may apply the claim to the vectors $v_j = Ae_j$ and conclude that $Ae_j = \lambda_j e_{\sigma(j)}$, $j = 1, \ldots, 4$. From this and (2.8) applied to $z = \chi = e_j$, we have $Bh_{u'}(e_j, e_j) = |\lambda_j|^2 h_u(e_{\sigma(j)}, e_{\sigma(j)})$, $j = 1, \ldots, 4$. Hence the real 2×2 invertible matrix B maps each vector in the set $S_{u'} := \{(1, 0), (1, 1), (1, u'), (0, 1)\}$ into a positive multiple of a vector in the set $S_u := \{(1, 0), (1, 1), (1, u), (0, 1)\}$. Since each of the vectors $(1, 1)$ and $(1, u')$ is a linear combination with positive coefficients of the vectors $(1, 0)$ and $(0, 1)$, the same is true of their images under the linear map B. It follows that each of the vectors $(1, 0)$ and $(0, 1)$ is mapped into a positive multiple of one of the same two vectors. Similar reasoning shows that up to positive scalar multiples B sends the vectors in the set $S_{u'}$ into those of S_u either by preserving the order of these vectors or reversing it. In either case, a simple calculation shows that necessarily $u = u'$. □

As a consequence of Lemma 2.4, it follows that for any $u, u' \in \mathbb{R}$, $u, u' > 1$ and $p \in M_u$, $p' \in M_{u'}$ the germs (M_u, p) and $(M_{u'}, p')$ are equivalent if and only if $u = u'$. Indeed since $(M_u, p) \sim (M_u, 0)$ for any $p \in M_u$ and any $u \in \mathbb{R}$, it suffices to assume $p = p' = 0$. If $(M_u, 0)$ and $(M_{u'}, 0)$ are (biholomorphically) equivalent, their Levi forms h_u and $h_{u'}$ are linearly equivalent, i.e., they must satisfy (2.8). Hence $u = u'$ by the lemma. Moreover, for any $u \in \mathbb{R}$ and any $p \in M_u$, the reader can easily check that M_u is of finite type at p. We may now give an example of a manifold which cannot be "flattened" at any point, as announced above.

Example 2.8. Let $h_u(z, z)$ be defined by (2.7). Consider the generic submanifold $M \subset \mathbb{C}^7$ of codimension 3 given in the coordinates $(z, w, u) \in \mathbb{C}^4 \times \mathbb{C}^2 \times \mathbb{C}$ by

$$\operatorname{Im} u = 0, \quad \operatorname{Im} w = h_u(z, z).$$

Then the CR orbits of M are $M \cap (\mathbb{C}^6 \times \{u\})$, $u \in \mathbb{R}$. By the observation preceding this example, the CR orbits of M are pairwise nonequivalent for $u > 1$. Hence, for any $q = (p, u) \in M$, with $p \in M_u$, $u > 1$, (M, q) is not equivalent to any product $(M' \times \mathbb{R}, 0)$ with $M' \subset \mathbb{C}^6$ a CR submanifold of finite type. Indeed, the CR orbit of $(0, u')$ in $M' \times \mathbb{R}$ is $M' \times \{u'\}$ and hence such orbits are equivalent to each other for different values of u'.

2.4. A structure result for points in general position. Putting Propositions 2.1, 2.2 and 2.3 together we obtain:

Theorem 2.5. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic submanifold. Then there exist integers $0 \le r_1, r_2, r_3 \le N$, such that for $p \in M$ in general position*

$$(M, p) \sim (\widetilde{M} \times \mathbb{C}^{r_2} \times \{0\}, 0), \quad \widetilde{M} \times \mathbb{C}^{r_2} \times \{0\} \subset (\mathbb{C}^r \times \mathbb{R}^{r_3}) \times \mathbb{C}^{r_2} \times \mathbb{C}^{r_1}, \quad (2.10)$$

where $r := N - r_1 - r_2 - r_3$, $\widetilde{M} \subset \mathbb{C}^r \times \mathbb{R}^{r_3}$ is a finitely nondegenerate generic submanifold of $\mathbb{C}^{r+r_3}$ through 0 such that for $u \in \mathbb{R}^{r_3}$ near 0, the point $(0, u)$ is in $\widetilde{M}$ and $\widetilde{M} \cap (\mathbb{C}^r \times \{u\})$ is a CR submanifold of finite type at $(0, u)$.

2.5. The hypersurface case. In the case $M \subset \mathbb{C}^N$ is a hypersurface, Theorem 2.5 can be reformulated in a simpler form. First, M is generic in $\mathbb{C}^N$, so that $r_1 = 0$. Also, as mentioned in §2.3, for $N \ge 2$, any hypersurface which is finitely nondegenerate at 0 is necessarily of finite type at 0, so that $r_3 = 0$. Hence we have

Corollary 2.6. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic hypersurface. Then exactly one of the following alternatives holds.*

(a) *There exists an integer $0 \le r_2 \le N - 2$ such that for all $p \in M$ in general position, $(M, p) \sim (\widetilde{M} \times \mathbb{C}^{r_2}, 0)$, where $\widetilde{M} \subset \mathbb{C}^{N-r_2}$ is a finitely nondegenerate hypersurface (and hence of finite type) through 0.*

(b) *For all $p \in M$ in general position, $(M, p) \sim (\mathbb{R} \times \mathbb{C}^{N-1}, 0)$.*

3. Finite nondegeneracy and higher order tensors for CR manifolds

Let $M \subset \mathbb{C}^N$ be a smooth CR submanifold, and $p \in M$. The following construction generalizes the Levi form as given by (2.2). For $s \ge 1$, consider the linear subspace $F_p^s \subset T_p^{0,1}M$ of all possible values $L(p)$ of a $(0, 1)$ vector field L such that (2.4) fails to hold for all $k < s$ and all $(0, 1)$ vector fields $L_1, \dots, L_k$. We obtain a decreasing sequence of subspaces

$$T_p^{0,1}M = F_p^1 \supset \cdots \supset F_p^s \supset \cdots .$$

Then there exists a unique multilinear map

$$\mathcal{L}_p^s : (T_p^{0,1}M)^s \times F_p^s \to (T_pM/T_p^cM) \otimes \mathbb{C}$$

satisfying

$$\mathcal{L}_p^s(L_1(p), \dots, L_s(p), L(p)) = \pi[L_1, \dots, [L_s, \overline{L}] \dots](p), \tag{3.1}$$

for all $(0, 1)$ vector fields $L, L_1, \dots, L_s$ with $L(p) \in F_p^s$, where π is as in (2.3). Indeed, it follows from the construction of the subspaces F_p^s that the right-hand side of (3.1) is multilinear in its arguments over the ring of smooth complex functions. The tensor $\mathcal{L}_p^s$ is complex-linear in the s first arguments and antilinear in the last one. For $s = 1$ we obtain a multiple of the Levi form; i.e., $\mathcal{L}_p^1(X_1, X_2) = 2i\mathcal{L}_p(X_1, X_2)$.

The tensors analogous to $\mathcal{L}_p^s$ were introduced by Ebenfelt [E 1998]. For $s = 2$, the definition of $\mathcal{L}_p^2$ is due to Webster [W 1995] (in the case M is a hypersurface), where $\mathcal{L}_p^2$ is called the *cubic form*. The subspaces F_p^s and the tensors $\mathcal{L}_p^s$ are also related to the submodules N_s, $1 \le s < \infty$, of the C^∞-module of all $(0, 1)$ vector fields defined by Freeman [F 1977] inductively as follows. Let N_1 be the module of all $(0, 1)$ vector fields on M. If N_{s-1} is defined, let $N_s \subset N_{s-1}$ be the C^∞-submodule consisting of all $(0, 1)$ vector fields L such that (2.4) fails to hold for all $k < s$, all $(0, 1)$ vector fields $L_1, \dots, L_k$ *and all* $p \in M$. (Actually, Freeman's submodules are the conjugates of the N_s's defined here.) If all subspaces F_q^s have constant dimension for q in a neighborhood of p, N_s consists precisely of their sections. However, in general, a subspace F_p^s may be nontrivial even if $N_s = 0$. For instance, this happens for $s = 2$ if the Levi form $\mathcal{L}_p$ is nondegenerate for p in general position but degenerate at some point.

Example 3.1. The hypersurface $M \subset \mathbb{C}^2$ given by $\operatorname{Im} w = |z|^4$ is finitely degenerate on $\{0\} \times \mathbb{R}$ and finitely nondegenerate outside this line. For $p = 0$ we have $F_0^s = T_0^{0,1}M$ for all $s \ge 1$. On the other hand, the submodules N_s are all zero for $s \ge 2$ and thus "don't notice" the degeneracy.

We have the following characterization of k-nondegeneracy:

Proposition 3.1. *If $M \subset \mathbb{C}^N$ is a smooth CR submanifold and $p \in M$, then the following are equivalent for $k \ge 1$:*

(i) *M is k-nondegenerate at p,*

(ii) *k is the smallest integer for which $F_p^{k+1} = 0$,*

(iii) $F_p^k \neq \{0\}$ *and for any* $X \in F_p^k$, $X \neq 0$, *there exist* $X_1, \ldots, X_k \in T_p^{0,1}M$ *with* $\mathcal{L}_p^k(X_1, \ldots, X_k, X) \neq 0$.

We now give an equivalent definition of finite nondegeneracy in terms of a defining function of M. If $M \subset \mathbb{C}^N$ is a smooth CR submanifold of codimension d and $p \in M$, a (smooth) *defining function* for M near p is a real smooth map $\rho = (\rho^1, \ldots, \rho^d)$ of rank d defined in a neighborhood of p in $\mathbb{C}^N$ such that M is locally defined near p by $\rho(Z, \overline{Z}) = 0$. We write $\rho_Z^j = (\rho_{Z_1}^j, \ldots, \rho_{Z_N}^j)$ for the complex gradient of ρ^j, $1 \leq j \leq d$. We have the following.

Proposition 3.2. *Let $M \subset \mathbb{C}^N$ be a smooth CR submanifold and ρ a smooth defining function of M near $p \in M$. Then for an integer $k \geq 0$, the following are equivalent.*

(i) *M is k-nondegenerate at p.*

(ii) *The integer k is minimal such that the collection of vectors $(L_1 \ldots L_s \rho_Z^j)(p, \bar{p})$ span $\mathbb{C}^N$ for $1 \leq j \leq d$, $0 \leq s \leq k$, and all choices of* $(0, 1)$ *vector fields $L_1, \ldots, L_s$ on M.*

As mentioned in Remark 2.3 above, k-nondegeneracy at p can be checked using any local basis of $(0, 1)$ vector fields on M. Also, one can directly check that condition (ii) of Proposition 3.2 does not depend on the choice of the defining function ρ and that in this condition, if suffices to take the vector fields $L_1, \ldots, L_s$ from a fixed local basis of $(0, 1)$ vector fields on M near p. To check the equivalence of (i) and (ii) in the case of a generic submanifold, the reader can take "normal" coordinates as in **[BER 1999a]** and do the calculation using the basis of vector fields given by (11.2.18) in **[BER 1999a]** and the defining function ρ given there. (A related calculation is done in the reference cited here.)

4. Different notions of equivalences for real submanifolds

4.1. Formal equivalence. In practice, biholomorphic equivalence of two germs of real submanifolds in $\mathbb{C}^N$ can be hard to check. For instance, in the work of Chern–Moser-**[CM 1974]** the so-called *formal equivalence* is established first and then the convergence of the formal map is proved, yielding biholomorphic equivalence.

Definition 4.1. A *formal equivalence* between two germs, (M, p) and (M', p'), of real-analytic submanifolds in $\mathbb{C}^N$ is an invertible formal power series map

$$H(Z) = p' + \sum_{|\alpha| \geq 1} a_\alpha (Z - p)^\alpha, \quad a_\alpha \in \mathbb{C}^N, \quad Z = (Z_1, \ldots, Z_N), \tag{4.1}$$

sending M into M' in the "formal sense", i.e., satisfying

$$\rho'\big(H(Z(x)), \overline{H(Z(x))}\big) = 0 \tag{4.2}$$

for some real-analytic parametrization $x \mapsto Z(x)$ of M near $p = Z(0)$ and some real-analytic defining function $\rho'(Z, \overline{Z})$ of M' near p'.

Equality in (4.2) is understood in the sense of formal power series in x. The formal map H given by (4.1) is invertible if the vectors $\{\partial H/\partial Z_j(p), 1 \leq j \leq N\}$ are linearly independent in $\mathbb{C}^N$. It is not hard to see that if (4.2) holds for some choice of parametrization $Z(x)$ of M and defining function ρ' of M', then it holds for any other choice. We shall say that the germs (M, p) and (M', p') are *formally equivalent* if there exists a formal equivalence between them. Formal equivalence of (M, p) and (M', p') translates into the existence of solutions for an infinite system of polynomial equations which the coefficients of H must satisfy. Formal equivalence is often easier to check than biholomorphic equivalence. One of the main results in **[BRZ 2000]** (see Theorem 6.1 below) states that for points in general position these two notions of equivalence coincide. Moreover, it is shown that any given formal equivalence can be "corrected" to a convergent one without changing terms of a prescribed finite order.

The assumption that the point p be in general position cannot be dropped. Indeed, in $\mathbb{C}^2$ there exist a pair of germs of formally, but not biholomorphically, equivalent 2-dimensional real-analytic (non CR) submanifolds. Moser and Webster proved in **[MW 1983]**, Proposition 6.1, that no neighborhood of 0 in the 2-dimensional (non CR) submanifold $M \subset \mathbb{C}^2$ given by

$$w = |z|^2 + \gamma\bar{z}^2 + \gamma z^3\bar{z}$$

can be biholomorphically transformed into the hyperplane $\mathbb{C} \times \mathbb{R}$ for $\gamma > 1/2$. On the other hand, they show that M is formally equivalent to a 2-dimensional real-analytic submanifold contained in $\mathbb{C} \times \mathbb{R}$ provided γ is not exceptional, i.e., if $(1/\pi)\arccos(1/2\gamma)$ is not a rational number. The authors of the present paper are not aware of any example of pairs of germs of real-analytic CR submanifolds that are formally but not biholomorphically equivalent.

4.2. CR equivalence and k-equivalence. A *CR function* on a smooth CR submanifold $M \subset \mathbb{C}^N$ is a smooth (C^∞) complex-valued function defined on M satisfying the Cauchy–Riemann equations restricted to M. More precisely, f is CR on M if $Lf \equiv 0$ for every $(0, 1)$ vector field L on M. If M and M' are germs of smooth CR submanifolds of $\mathbb{C}^N$ at p and p' respectively, we say that (M, p) is *CR equivalent* to (M', p') if there is a CR diffeomorphism (a diffeomorphism whose components are CR functions) between open neighborhoods of p in M and of p' in M' respectively and taking p to p'. Such a diffeomorphism is called a *CR equivalence.*

It is known that if f is a CR function defined in a neighborhood of p in M, then there is a formal (holomorphic) power series $\sum_\alpha c_\alpha(Z-p)^\alpha$, $Z = (Z_1, \ldots, Z_N)$, $c_\alpha \in \mathbb{C}$, whose restriction to M coincides with the Taylor series of f at p. Moreover, if M is generic, then such a formal power series is unique. (See e.g. [BER 1999a], Proposition 1.7.14 for the generic case.) It follows that if h is a CR equivalence between two real-analytic germs (M, p) and (M', p') of CR submanifolds of $\mathbb{C}^N$, then the corresponding vector-valued formal power series H of the form (4.1) (obtained from the components of h by the Taylor series property of CR functions mentioned above) satisfies (4.2) and can be assumed to be invertible; hence H is a formal equivalence between (M, p) and (M', p'). On the other hand, the restriction to M of a biholomorphic equivalence between (M, p) and (M', p') is obviously a CR equivalence. Thus the notion of CR equivalence lies between that of formal and biholomorphic equivalence:

$$\text{biholomorphic equivalence} \implies \text{CR equivalence} \implies \text{formal equivalence.}$$

A weaker notion than that of formal equivalence is that of k-equivalence for an integer $k > 1$.

Definition 4.2. For an integer $k > 1$ we say that two germs, (M, p) and (M', p'), of real-analytic submanifolds of $\mathbb{C}^N$ are *k-equivalent* if there exists a biholomorphic map H between neighborhoods of p and p' in $\mathbb{C}^N$, with $H(p) = p'$, such that

$$\rho'\big(H(Z(x)), \overline{H(Z(x))}\big) = O(|x|^k)$$

for some real-analytic parametrization $x \mapsto Z(x)$ of M near $p = Z(0)$ and some real-analytic defining function $\rho'(Z, \overline{Z})$ of M' near p'. Such an H is called a *k-equivalence* between (M, p) and (M', p').

Again here the definition of k-equivalence is independent of the choice of the parametrization $Z(x)$ of M and of the choice of the defining function ρ' of M'. We also note that if H is a k-equivalence (or a formal equivalence), by taking its Taylor polynomial of order $k - 1$, we can find another k-equivalence whose components are polynomials. Hence in Definition 4.2 we could have assumed that H is a biholomorphism with polynomial components. Similarly, we could have also assumed that H is just a formal invertible mapping, rather than a biholomorphism. Then any formal equivalence may be considered as a k-equivalence for every k.

Example 4.1. If $k > 0$ is an integer, then the identity map is a $2k$-equivalence between the germs at 0 of the real hyperplane $M := \mathbb{C} \times \mathbb{R} \subset \mathbb{C}^2$ and the hypersurface M' given by $\operatorname{Im} w = |z|^{2k}$. However, it is easily checked that there is no formal equivalence between $(M, 0)$ and $(M', 0)$.

The example shows that even very different looking germs of submanifolds can be k-equivalent for some fixed k without being formally equivalent. The situation becomes rather different if we require (M, p) and (M', p') to be k-equivalent for every k. This means the existence of a sequence of biholomorphic maps H_k each sending (M, p) into (M', p') up to order k, as in Definition 4.2. In particular, as noted above, formal equivalence implies the existence of such a sequence. On the other hand, given a sequence of k-equivalences, in general, one cannot put them together to obtain a formal equivalence. Nevertheless, for points in general position, the main result in **[BRZ 2000]** states that the existence of such a sequence implies that (M, p) and (M', p') are formally and even biholomorphically equivalent. The authors of the present paper are not aware of any example of pairs of germs (M, p) and (M', p') of real analytic submanifolds in $\mathbb{C}^N$ which are k-equivalent for every $k > 1$, but not formally equivalent.

5. Structure decompositions and k-equivalences

In this section we consider the extent to which the decompositions given by Propositions 2.1, 2.2 and 2.3 (and summarized in Theorem 2.5) are invariant under different notions of equivalences.

We first consider invariance under biholomorphic equivalences. We already remarked that the numbers r_1, r_2, r_3 introduced in §2 are biholomorphic invariants. Write $r := N - r_1 - r_2 - r_3$ for brevity. Now assume that we have two germs at 0 of real-analytic submanifolds in $\mathbb{C}^N$ of the form $\widetilde{M} \times \mathbb{C}^{r_2} \times \{0\}$ and $\widetilde{M}' \times \mathbb{C}^{r_2} \times \{0\}$ with $\widetilde{M}, \widetilde{M}' \subset \mathbb{C}^r \times \mathbb{R}^{r_3}$ as in Theorem 2.5. That is, both $\widetilde{M}$ and $\widetilde{M}'$ are finitely nondegenerate generic submanifolds through 0 containing all points of the form $(0, u)$ for $u \in \mathbb{R}^{r_3}$ near 0, and such that $\widetilde{M} \cap (\mathbb{C}^r \times \{u\})$ and $\widetilde{M}' \cap (\mathbb{C}^r \times \{u\})$ are of finite type for u small. We fix local holomorphic coordinates $Z = (Z^0, Z^3, Z^2, Z^1) \in \mathbb{C}^r \times \mathbb{C}^{r_3} \times \mathbb{C}^{r_2} \times \mathbb{C}^{r_1}$ near 0 and similarly we write $H = (H^0, H^3, H^2, H^1)$ for the components of a $\mathbb{C}^N$-valued map H.

Proposition 5.1. *Let $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be a biholomorphic equivalence between the germs at 0 of $\widetilde{M} \times \mathbb{C}^{r_2} \times \{0\}$ and $\widetilde{M}' \times \mathbb{C}^{r_2} \times \{0\}$. Then we have:*

(i) $H^1(Z^0, Z^3, Z^2, 0) \equiv 0$, *i.e., H sends $\mathbb{C}^r \times \mathbb{C}^{r_3} \times \mathbb{C}^{r_2} \times \{0\}$ into itself;*

(ii) $\frac{\partial H^0}{\partial Z^2}(Z^0, Z^3, Z^2, 0) \equiv 0$, $\frac{\partial H^3}{\partial Z^2}(Z^0, Z^3, Z^2, 0) \equiv 0$, *i.e., H preserves the affine subspaces given by*

$$Z^1 = 0, \quad (Z^0, Z^3) = \text{const};$$

(iii) $\frac{\partial H^3}{\partial Z^0}(Z^0, Z^3, Z^2, 0) \equiv 0$, *i.e., H also preserves the affine subspaces given by*

$$Z^1 = 0, \quad Z^3 = \text{const}.$$

In particular, the restriction of (H^0, H^3) to $\mathbb{C}^r \times \mathbb{C}^{r_3} \times \{0\} \times \{0\}$ is a biholomorphic equivalence between $(\widetilde{M}, 0)$ and $(\widetilde{M}', 0)$.

Remark 5.1. Proposition 5.1 can be reformulated "geometrically" as follows. *A biholomorphic equivalence between real-analytic submanifolds preserves their intrinsic complexifications, maximal tangent holomorphic foliations and CR orbits.*

A statement similar to Proposition 5.1 also holds for formal equivalences. However, both statements for formal and biholomorphic equivalences are in fact special cases of more general invariance properties under k-equivalences. In the following proposition, as was illustrated by Example 4.1, it is crucial to require the existence of a k-equivalence for every k.

Proposition 5.2 ([BRZ 2000], Proposition 4.1). *Suppose that (M, p) and (M', p') are k-equivalent for all $k > 1$. Then the numbers r_1, r_2, r_3 in Theorem 2.5 for M coincide with the ones for M'.*

Proposition 5.3 ([BRZ 2000], Lemma 4.4, Lemma 5.3). *Under the assumptions of Proposition 5.1 suppose that $\widetilde{M}'$ is l-nondegenerate and let H be a k-equivalence between $\widetilde{M} \times \mathbb{C}^{r_2} \times \{0\}$ and $\widetilde{M}' \times \mathbb{C}^{r_2} \times \{0\}$. Then we have:*

(i) $H^1(Z^0, Z^3, Z^2, 0) = O(|Z|^k)$;

(ii) $\frac{\partial H^0}{\partial Z^2}(Z^0, Z^3, Z^2, 0) = O(|Z|^{k-l-1})$, $\frac{\partial H^3}{\partial Z^2}(Z^0, Z^3, Z^2, 0) = O(|Z|^{k-l-1})$ *provided $k > l$;*

In particular, the restriction of (H^0, H^3) to $\mathbb{C}^r \times \mathbb{C}^{r_3} \times \{0\} \times \{0\}$ is a k-equivalence between $(\widetilde{M}, 0)$ and $(\widetilde{M}', 0)$.

6. Comparison of different notions of equivalences

The following, which is one of the main results of **[BRZ 2000]**, states that the four notions of equivalence discussed above actually coincide at all points in general position.

Theorem 6.1 ([BRZ 2000], Corollary 14.1). *Let $M \subset \mathbb{C}^N$ be a connected real-analytic submanifold. Then for any $p \in M$ in general position and any germ (M', p') of a real-analytic submanifold in $\mathbb{C}^N$, the following conditions are equivalent:*

(i) *(M, p) and (M', p') are k-equivalent for all $k > 1$;*

(ii) *(M, p) and (M', p') are formally equivalent;*

(iii) *(M, p) and (M', p') are CR equivalent;*

(iv) *(M, p) and (M', p') are biholomorphically equivalent.*

As mentioned in §4, the implications (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) hold trivially. It was shown in **[BER 1999b]** that if M and M' are real-analytic generic submanifolds which are finitely nondegenerate and of finite type at p and p' respectively, then any formal equivalence H between (M, p) and (M', p') is necessarily convergent. In particular, one obtains the equivalence of conditions (ii), (iii) and (iv) for M a connected real-analytic generic submanifold which is finitely nondegenerate and of finite type at some point (and hence at all points in general position).

In the case that M is a real-analytic *hypersurface*, one can use the result in **[BER 1999b]** mentioned above together with Proposition 5.3 to prove the equivalence of (ii), (iii), and (iv) of Theorem 6.1 as follows. We begin with the structure theory, Corollary 2.6, for hypersurfaces at points in general position. Since the fact that (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) in case (b) of Corollary 2.6 can be easily proved, we may assume that condition (a) of that corollary holds. Hence we may assume that $(M, p) = (\widetilde{M} \times \mathbb{C}^{r_2}, 0)$ and that $(M', p') = (\widetilde{M}' \times \mathbb{C}^{r_2}, 0)$, where $\widetilde{M}$ and $\widetilde{M}'$ are finitely nondegenerate hypersurfaces and hence of finite type at 0. Let H be a formal equivalence between (M, p) and (M', p'). By Proposition 5.3 (here $r_1 = r_3 = 0$, $r = N - r_2$, and $H = (H^0, H^2)$), we conclude that the restriction $\widetilde{H}$ of H^0 to $\mathbb{C}^r \times \{0\}$ is a formal equivalence between $(\widetilde{M}, 0)$ and $(\widetilde{M}', 0)$. Since $\widetilde{M}$ and $\widetilde{M}'$ are finitely nondegenerate and of finite type at 0, it follows from the result in **[BER 1999b]** mentioned above that $\widetilde{H}$ must be already convergent. It is then easy to extend $\widetilde{H}$ to a holomorphic equivalence between $(\widetilde{M} \times \mathbb{C}^{r_2}, 0)$ and $(\widetilde{M}' \times \mathbb{C}^{r_2}, 0)$. In fact, one may choose such a biholomorphic equivalence in such a way that its Taylor series coincides with that of H up to any preassigned order.

For submanifolds of higher codimension it is not possible to reduce to the results of **[BER 1999b]**, even to prove that the notions of formal and biholomorphic equivalence coincide, since the submanifold $\widetilde{M}$ given in Theorem 2.5 need not be equivalent to a product of a CR manifold of finite type and $\mathbb{R}^m$ for some m. Indeed, if M is the finitely nondegenerate generic submanifold of codimension 3 in $\mathbb{C}^7$ given in Example 2.8, then $M = \widetilde{M}$, but $\widetilde{M}$ is not equivalent to such a product.

We now present some of the ideas involved in the proof of (ii) $\Longrightarrow$ (iv) in Theorem 6.1, as given in **[BRZ 2000]**, for a general real-analytic submanifold M. By making use of Theorem 2.5 and Proposition 5.1, we first reduce to the case where $p = p' = 0$, $M = \widetilde{M}$ and $M' = \widetilde{M}'$, with $\widetilde{M}$ and $\widetilde{M}'$ as in Theorem 2.5. (That is, we assume $r_1 = r_2 = 0$ in Theorem 2.5.)

The next step is to obtain a parametrization of all formal equivalences H between $(M, 0)$ and $(M', 0)$ by their (formal) jets along the linear subspace $C := \{0\} \times \mathbb{C}^{r_3} \subset \mathbb{C}^{N-r_3} \times \mathbb{C}^{r_3}$, which is transversal to the CR orbits of M. In the coordinates $(Z^0, u) \in \mathbb{C}^{N-r_3} \times \mathbb{C}^{r_3}$ the required parametrization has the form

$$H(Z^0, u) = \Gamma\big((\partial^\alpha H(0, u))_{|\alpha| \le k}, Z^0, u\big), \tag{6.1}$$

where k is a number depending only on the dimension N, Γ is a $\mathbb{C}^N$-valued holomorphic map defined in some neighborhood of $j^k H(0) \times \{0\}$ in the space $J^k(\mathbb{C}^N, \mathbb{C}^N) \times \mathbb{C}^{r_3}$, with $j^k H(0) = \big(\partial^\alpha H(0, 0)\big)_{|\alpha| \le k}$ and $J^k(\mathbb{C}^N, \mathbb{C}^N)$ denoting the space of k-jets at 0 of holomorphic maps from $\mathbb{C}^N$ to $\mathbb{C}^N$. Here it is important to note that Γ does not depend on the formal mapping H. Equality in (6.1) is in the sense of formal power series in Z^0 and u. The identity (6.1) is a simplified version of the statement of Theorem 10.1 in **[BRZ 2000]**.

The third step is to use (6.1) to obtain a system of holomorphic equations which must be satisfied by the formal series components of $(\partial^\alpha H(0,u))_{|\alpha|\le k}$. To this system we apply Artin's approximation theorem **[A 1968]** to conclude that there exists a convergent solution for this system of equations. This yields the existence of a holomorphic map $\widehat{H} : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$, which is the desired biholomorphic equivalence between $(M, 0)$ and $(M', 0)$. In fact, we can even choose $\widehat{H}$ in such a way that its Taylor series coincides with that of H up to a preassigned order.

7. General conditions for the convergence of formal equivalences

We shall give here a result about convergence of formal equivalences between two germs (M, p) and (M', p') for p and p' in general position, more general than that given in **[BER 1999b]** mentioned above. We restrict ourselves to the case where M and M' are generic submanifolds of $\mathbb{C}^N$, i.e., $r_1 = 0$. By making use of Theorem 2.5 we may assume that $(M, p) = (\widetilde{M} \times \mathbb{C}^{r_2}, 0)$ such that $\widetilde{M} \subset \mathbb{C}^{N-r_3}$ is as in Theorem 2.5 (but with $r_1 = 0$). Similarly we assume $(M', p') = (\widetilde{M}' \times \mathbb{C}^{r_2}, 0)$, with again $\widetilde{M}'$ as in Theorem 2.5, with the same integers r_2 and r_3. We shall assume (M, p) and $M', p')$ have this form for the remainder of this section.

Theorem 7.1 (**[BRZ 2000]**, Corollary 10.3). *Let (M, p) and (M', p') be as above. Then there exists an integer $\kappa \ge 0$ such that a formal equivalence H between (M, p) and (M', p') is convergent if and only if, for*

$$Z = (Z^0, Z^3, Z^2) \in \mathbb{C}^{N-r_2-r_3} \times \mathbb{C}^{r_3} \times \mathbb{C}^{r_2}, \quad H = (H^0, H^3, H^2),$$

both of the following conditions are satisfied.

(i) *All partial derivatives* $\frac{\partial^{|\alpha|}(H^0, H^3)}{\partial (Z^0)^\alpha}(0, Z^3, 0)$ *are convergent for* $|\alpha| \le \kappa$;

(ii) $H^2(Z)$ *is convergent.*

In fact, κ can be chosen to be $2(d+1)l$, where d is the codimension of $\widetilde{M}$ in $\mathbb{C}^{N-r_2}$ and l is chosen such that $\widetilde{M}$ is l-nondegenerate.

In the case where $r_2 = 0$, the proof of Theorem 7.1 follows immediately from the parametrization of all formal mappings given by (6.1), since in that case condition (i) of the theorem implies that the right hand side of (6.1) is convergent and hence so is the left hand side, $H(Z)$. For the general case where r_2 need not be 0, one also needs to use Proposition 5.3.

8. Real-algebraic submanifolds

A submanifold $M \subset \mathbb{C}^N$ is *real-algebraic* if it is contained in a real-algebraic subvariety of the same dimension. The basic example is given by the sphere $\sum_{j=1}^N |z_j|^2 = 1$. Most examples of real submanifolds included in this article are real-algebraic. The study of local biholomorphic maps sending pieces of spheres into each other goes back to Poincaré **[P 1907]** and Tanaka **[T 1962]**. Webster proved in **[W 1977]** that local biholomorphic maps sending open pieces of Levi-nondegenerate real-algebraic hypersurfaces into each other are complex-algebraic, i.e., their graphs are contained in complex-algebraic subvarieties of the same dimension. The algebraic properties of holomorphic maps sending one real-algebraic submanifold into another reveal the optimal nondegeneracy conditions for points in general position and have been intensively studied (see **[S 1991]**, **[H 1994]**, **[BR 1995]**, **[S 1995]**, **[Z 1995]**, **[BER 1996]**, **[SS 1996]**, **[Mi 1998]**, **[CMS 1999]**, **[Z 1999]**).

If M is a connected real-algebraic submanifold of $\mathbb{C}^N$, we say that a property holds for points in *general algebraic position* if it holds for all $p \in M$ outside a proper real-algebraic subvariety of M. Also, a stronger notion of equivalence, that of algebraic equivalence, can be naturally considered. Two germs of real-analytic submanifolds of $\mathbb{C}^N$, (M, p) and (M', p'), are said to be *algebraically equivalent* if there exists a biholomorphic equivalence between them which is complex-algebraic. Then the analogues of Propositions 2.1, 2.2, 2.3 and hence of Theorem 2.5 also hold in the category of real-algebraic submanifolds and algebraic equivalences for points in general algebraic position. The proof is based on the algebraic version of the implicit function theorem and other elementary properties (see e.g. **[BER 1999a]**, §5.4). In particular, the algebraic analogue of Proposition 2.1 follows from **[BER 1999a]**, Proposition 5.4.3 (d). The algebraic analogue of Proposition 2.2 can be obtained by repeating the proof of Proposition 3.1 in **[BRZ 2000]**. In contrast to this, the argument of the proof of Proposition 3.3 in **[BRZ 2000]** cannot be directly adapted to the algebraic case since the algebraic version of the Frobenius theorem does not hold. The algebraic analogue of Proposition 2.3 was shown in **[BER 1996]** (Lemma 3.4.1) by using the Segre sets rather than the Frobenius theorem. In particular, the CR orbits of real-algebraic CR submanifolds are algebraic (**[BER 1996]**, Corollary 2.2.5), whereas the orbits of single vector fields in T^cM (the real parts of (0, 1) vector fields) need not be algebraic.

Example 8.1. Consider the real-algebraic hypersurface $M \subset \mathbb{C}^2$ given in real coordinates by $y_2 = x_2 y_1$ where $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2)$. The sections of the complex tangent subbundle T^cM are spanned at each point by the vector fields

$$X = x_2 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial y_1}, \quad Y = JX_1 = x_2 \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial x_1}.$$

The integral curve C for X through $p_0 = (x_1^0, x_2^0, y_1^0, y_2^0) \in M$ is given by

$$x_2 = x_2^0 e^{x_1 - x_1^0}, \quad y_1 = y_1^0 e^{x_1^0 - x_1}, \quad y_2 = y_2^0.$$

Hence C is not algebraic if $x_2^0 \neq 0$ or $y_1^0 \neq 0$. (In contrast, the orbits of Y are all algebraic.) However, the CR orbit of any point $p_0 = (x_1^0, x_2^0, y_1^0, y_2^0) \in M$ with $x_2^0 \neq 0$ is (M, p_0), while when $x_2^0 = 0$ (and hence $y_2^0 = 0$) the CR orbit is $(\mathbb{C} \times \{0\}, p_0)$. Thus the algebraicity of these CR orbits cannot be proved by showing the algebraicity of the integral curves of the basis of sections of T^cM given by X, Y.

9. Algebraic equivalence for real-algebraic submanifolds

The following strengthens Theorem 6.1 in the case of real-algebraic submanifolds.

Theorem 9.1. *Let $M \subset \mathbb{C}^N$ be a connected real-algebraic submanifold. Then for any $p \in M$ in general algebraic position and any germ (M', p') of a real-algebraic submanifold in $\mathbb{C}^N$, the equivalent conditions* (i)–(iv) *of Theorem* 6.1 *are also equivalent to*

(v) *(M, p) and (M', p') are algebraically equivalent.*

The proof of Theorem 9.1 can be obtained by adapting the proof of Theorem 6.1 given in **[BRZ 2000]** to the algebraic case. It is sufficient to prove that (iv) implies (v). As in the proof of Theorem 6.1 the first step is to reduce the general situation by a complex-algebraic change of coordinates to the case where $p = p' = 0$, $M = \widetilde{M}$ and $M' = \widetilde{M}'$ with $\widetilde{M}$ and $\widetilde{M}'$ as in Theorem 2.5, or rather its real-algebraic analogue mentioned above. Here again, $r_1 = r_2 = 0$. The second step is to obtain an algebraic parametrization of all biholomorphic equivalences H between $(M, 0)$ and $(M', 0)$ by their jets along the linear subspace $C = \{0\} \times \mathbb{C}^{r_3}$. This parametrization takes the form (6.1) with Γ a map which is not only holomorphic but also complex-algebraic, defined in some neighborhood of $j^kH(0) \times \{0\}$ in the jet space $J^k(\mathbb{C}^N, \mathbb{C}^N) \times \mathbb{C}^{r_3}$. Note that in this case the right hand side of (6.1) is a convergent power series in (Z_0, u). If $u \mapsto \left(\partial^\alpha H(0, u)\right)_{|\alpha| \le k}$ is complex algebraic, then the algebraicity of H follows immediately from the analogue of (6.1). Hence in the special case when M is of finite type, i.e., $r_3 = 0$, H is parametrized by the single jet $j^kH(0) = \left(\partial^\alpha H(0)\right)_{|\alpha| \le k}$ so that the biholomorphic equivalence H is algebraic itself. If $r_3 > 0$, the jet $\left(\partial^\alpha H(0, u)\right)_{|\alpha| \le k}$ and hence H need not be algebraic. Then we have to modify $\left(\partial^\alpha H(0, u)\right)_{|\alpha| \le k}$ to obtain an algebraic equivalence. This is done by using an algebraic version of Artin's approximation theorem **[A 1969]**. Here again we can choose the Taylor series of the algebraic equivalence to coincide with that of H up to a preassigned order.

Example 9.1. Let $M_0 \subset \mathbb{C}^{N-1}$ be an arbitrary finitely nondegenerate real-algebraic submanifold through 0 and set $M = M' := M_0 \times \mathbb{R} \subset \mathbb{C}^N$. Then both M and M' are finitely nondegenerate of infinite type and any biholomorphic map $H(z, w) := (z, \phi(w))$ with $\phi\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ sending $\mathbb{R}$ into itself is a biholomorphic equivalence between $(M, 0)$ and $(M', 0)$. It is clear that if ϕ is not algebraic, then H is not algebraic. On the other hand, the identity mapping is an algebraic equivalence between $(M, 0)$, and $(M', 0)$, and there are many others.

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