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Projection on Segre varieties and determination of holomorphic mappings between real submanifolds

Dedicated to Professor Sheng GONG on the occasion of his 75th birthday

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Abstract It is shown that a germ of a holomorphic mapping sending a real-analytic generic submanifold of finite type into another is determined by its projection on the Segre variety of the target manifold. A necessary and sufficient condition is given for a germ of a mapping into the Segre variety of the target manifold to be the projection of a holomorphic mapping sending the source manifold into the target. An application to the biholomorphic equivalence problem is also given.

Keywords: Segre variety, generic submanifold, Segre mappings, finite type, biholomorphic equivalence.

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1 Introduction

In this paper, we show that a germ at 0 of a holomorphic mapping $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ sending one real-analytic generic submanifold $M \subset \mathbb{C}^N$ of finite type at 0 into another real-analytic generic submanifold $M' \subset \mathbb{C}^{N'}$ is determined by its projection onto Σ'_0, the Segre variety of M' at 0 (Theorem 1.1). We also give a necessary and sufficient condition for a germ at 0 of a holomorphic mapping $F: (\mathbb{C}^N, 0) \to (\Sigma'_0, 0)$ to be the projection of such a mapping H (Theorem 1.6). As a corollary, we obtain a criterion for two real-analytic generic submanifolds M, M' of finite type at $0 \in \mathbb{C}^N$ to be locally biholomorphically equivalent at 0 (Corollary 1.7). The main tools used in the proofs are the iterated Segre mappings, as previously introduced by the authors in refs. [1,2], and also a new invariant description of normal coordinates (Theorem 2.1), which may be of independent interest. Segre variety techniques in the context of mappings between real hypersurfaces were introduced in refs. [3, 4].

Let M be a real-analytic generic submanifold of codimension d in $\mathbb{C}^N$ with $0 \in M$, given locally near 0 by

$$\rho_1(Z,\bar{Z}) = \cdots = \rho_d(Z,\bar{Z}) = 0, \tag{1.1}$$

where $\rho(Z,\zeta) := (\rho_1(Z,\zeta),\cdots,\rho_d(Z,\zeta))$ is a $\mathbb{C}^d$-valued holomorphic function such that $\partial_Z\rho_1 \wedge \cdots \wedge \partial_Z\rho_d \neq 0$ near 0 and

$$\rho(Z,\zeta) = \overline{\rho(\bar{\zeta},\bar{Z})}. \tag{1.2}$$

The generic submanifold M is said to be of finite type at p (in the sense of Kohn[5] and Bloom-Graham[6]) if the (complex) Lie algebra $\mathfrak{g}_M$ generated by all smooth $(1,0)$ and $(0,1)$ vector fields tangent to M satisfies $\mathfrak{g}_M(p) = \mathbb{C}T_pM$, where $\mathbb{C}T_pM$ is the complexified tangent space to M at p.

Let $U \subset \mathbb{C}^N$ be a sufficiently small open neighborhood of 0. For $p \in \mathbb{C}^N$ sufficiently close to 0, we denote by Σ_p the Segre variety of M at p defined by

$$\Sigma_p := \{Z \in U: \rho(Z,\bar{p}) = 0\}. \tag{1.3}$$

We observe, for future reference, that Σ_p is an n-dimensional complex submanifold of U, with $n = N - d$, for all such p. Moreover, it follows from (1.2) that $p \in \Sigma_p$ if and only p is in M, and also that $q \in \Sigma_p$ if and only if $p \in \Sigma_q$.

Theorem 1.1. Let M be a real-analytic generic submanifold of codimension d in $\mathbb{C}^N$ and of finite type at $0 \in M$. Then, for every $\lambda \in \mathbb{C}$, $0 < |\lambda| < 1$, there exist $2d+1$ germs at 0 of holomorphic functions $g_1^\lambda,\dots,g_d^\lambda$, $h_1^\lambda,\dots,h_{d+1}^\lambda: (\mathbb{C}^N,0) \to \mathbb{C}^N$, depending holomorphically on λ, such that $g_j^\lambda(0) \to 0$, $h_j^\lambda(0) \to 0$ as $\lambda \to 0$ and such that the following holds. If $M' \subset \mathbb{C}^{N'}$ is a real-analytic generic submanifold of codimension d' through 0 and $\tilde{z}$ a germ at 0 of a holomorphic submersion $\tilde{z}: (\mathbb{C}^{N'},0) \to (\Sigma_0',0)$, where Σ_0' is the Segre variety of M' at 0, such that $\tilde{z}^{-1}(0)$ is transversal to Σ_0', then there exists a germ at 0 of a holomorphic mapping $\Phi: ((\Sigma_0')^{2(d+1)},0) \to (\mathbb{C}^{N'},0)$ satisfying the following. If $H: (\mathbb{C}^N,0) \to (\mathbb{C}^{N'},0)$ is a germ at 0 of a holomorphic mapping such that $H(M) \subset M'$, then

$$H = \Phi \circ (\overline{\tilde{z}\circ H}\circ h_1^\lambda, \tilde{z}\circ H\circ g_1^\lambda,\dots,\overline{\tilde{z}\circ H}\circ h_d^\lambda, \tilde{z}\circ H\circ g_d^\lambda, \overline{\tilde{z}\circ H}\circ h_{d+1}^\lambda, \tilde{z}\circ H), \tag{1.4}$$

for all λ sufficiently small.

Remark 1.2. It follows from the proof of Theorem 1.1 that there exists an integer $l \geqslant 0$ such that each of the functions $g_1^\lambda,\dots,g_d^\lambda$, $h_1^\lambda,\dots,h_{d+1}^\lambda$ is given by a convergent power series of the form

$$a_0(\lambda) + \sum_{\alpha\in\mathbb{Z}_+^N\setminus\{0\}} a_\alpha(\lambda)\frac{Z^\alpha}{\lambda^{l|\alpha|}}, \tag{1.5}$$

where the coefficients $a_0(\lambda), a_\alpha(\lambda)$ are holomorphic in the unit disk $\mathbb{D}$, $a_0(0) = 0$. Another way of expressing this is saying that each of the functions $g_1^\lambda,\dots,g_d^\lambda$, $h_1^\lambda,\dots,h_{d+1}^\lambda$ is given by

$$g_j^\lambda(Z) = \hat{g}_j\left(\frac{Z}{\lambda^l},\lambda\right), \quad h_j^\lambda(Z) = \hat{h}_j\left(\frac{Z}{\lambda^l},\lambda\right), \tag{1.6}$$

where the $\hat{g}_1,\dots,\hat{g}_d$, $\hat{h}_1,\dots,\hat{h}_{d+1}$ are germs at 0 of holomorphic functions $(\mathbb{C}^N\times\mathbb{C},0) \to (\mathbb{C}^N,0)$.

Remark 1.3. Suppose that $P\colon (\mathbb{C}^{N'},0) \to (\Sigma'_0,0)$ is a germ at 0 of a holomorphic projection, i.e. $P|_{\Sigma'_0}$ is the identity on Σ'_0. The reader can easily verify that $\tilde{z} := P$ is a holomorphic submersion satisfying the assumptions of Theorem 1.1. Conversely, any holomorphic submersion $\tilde{z}$ as in Theorem 1.1 is a projection up to a local biholomorphism of Σ'_0 at 0.

An immediate corollary of Theorem 1.1 is the following.

Corollary 1.4. Let M, M', and $\tilde{z}$ be as in Theorem 1.1. If $H\colon (\mathbb{C}^N,0) \to (\mathbb{C}^{N'},0)$ is a germ at 0 of a holomorphic mapping such that $H(M) \subset M'$, then H is uniquely determined by $\tilde{z} \circ H$.

An algebraic reformulation of Corollary 1.4 can be given as follows. For any complex manifold X and $p \in X$, let $\mathcal{O}_X(p)$ denote the ring of germs at p of holomorphic functions on X. For $X = \mathbb{C}^N$, we write $\mathcal{O}_N(p)$ instead of $\mathcal{O}_{\mathbb{C}^N}(p)$. Recall that if $Y \subset X$ is a complex analytic subvariety through p, then the ring $\mathcal{O}_Y(p)$ of germs at p of holomorphic functions on Y is given by $\mathcal{O}_X(p)/I(Y)$, where $I(Y)$ denotes the ideal of germs vanishing on Y. Let H be a germ at p of a holomorphic mapping $(X,p) \to (W,q)$, where X and W are complex manifolds. The mapping H induces a ring homomorphism $\Phi_H\colon \mathcal{O}_W(q) \to \mathcal{O}_X(p)$, given by $\Phi_H(f) = f \circ H$ for $f \in \mathcal{O}_W(q)$. The reader can verify that the following result is a reformulation of Corollary 1.4.

Theorem 1.5. Let M, M', and Σ'_0 be as in Theorem 1.1 and denote by π the canonical homomorphism $\pi\colon \mathcal{O}_{N'}(0) \to \mathcal{O}_{\Sigma'_0}(0)$. Let $\phi\colon \mathcal{O}_{\Sigma'_0}(0) \to \mathcal{O}_{N'}(0)$ be any ring homomorphism such that $\pi \circ \phi\colon \mathcal{O}_{\Sigma'_0}(0) \to \mathcal{O}_{\Sigma'_0}(0)$ is an isomorphism. Then, for any $H\colon (\mathbb{C}^N,0) \to (\mathbb{C}^{N'},0)$ a germ at 0 of a holomorphic mapping such that $H(M) \subset M'$, the induced homomorphism Φ_H is uniquely determined by $\Phi_H \circ \phi$.

We give now a necessary and sufficient condition for a germ at 0 of a holomorphic mapping $F\colon (\mathbb{C}^N,0) \to (\Sigma'_0,0)$ to be of the form $\tilde{z} \circ H$, for some $\tilde{z}$ as in Theorem 1.1 and a holomorphic mapping H sending M into M'.

Theorem 1.6. Let M and d be as in Theorem 1.1. Then there exists an open, connected subset $\Omega \subset \mathbb{C}^N \times \mathbb{C}^{2(d+1)(N-d)-N}$ such that the set

$$\{(Z,\xi) \in \mathbb{C}^N \times \mathbb{C}^{2(d+1)(N-d)-N} \colon (Z,\xi) \in \Omega,\ Z = 0\},$$

is open in $\{0\} \times \mathbb{C}^{2(d+1)(N-d)-N}$ with 0 in its closure, and $2d+1$ germs at 0 of holomorphic mappings

$$A_1, \ldots, A_d, B_1, \ldots, B_{d+1}\colon (\Omega,0) \to \mathbb{C}^N$$

such that $A_j(0,\xi) \to 0$, $B_j(0,\xi) \to 0$ as $\xi \to 0$ (for $(0,\xi) \in \Omega$) and such that the following holds. If M', d', Σ'_0 are as in Theorem 1.1, then there exists a germ at 0 of a holomorphic mapping $\Psi\colon ((\Sigma'_0)^{2(d+1)},0) \to (\mathbb{C}^{d'},0)$ satisfying the following. Let $F\colon (\mathbb{C}^N,0) \to (\Sigma'_0,0)$ be a germ at 0 of a holomorphic mapping. If there exists a germ at 0 of a holomorphic mapping $H\colon (\mathbb{C}^N,0) \to (\mathbb{C}^{N'},0)$ and a germ at 0 of a holomorphic submersion $\tilde{z}\colon (\mathbb{C}^{N'},0) \to (\Sigma'_0,0)$ with $\tilde{z}^{-1}(0)$ transversal to Σ'_0 such that

$$H(M) \subset M', \quad F = \tilde{z} \circ H \tag{1.7}$$

then

$$\frac{\partial}{\partial\xi}\Psi(\overline{F}\circ B_1(Z,\xi), F\circ A_1(Z,\xi),\dots,\overline{F}\circ B_d(Z,\xi),$$
$$F\circ A_d(Z,\xi), \overline{F}\circ B_{d+1}(Z,\xi), F(Z))\equiv 0. \tag{1.8}$$

Conversely, suppose (1.8) holds. Then for every germ at 0 of a holomorphic submersion $\tilde{z}\colon(\mathbb{C}^{N'},0)\to(\Sigma_0',0)$ with $\tilde{z}^{-1}(0)$ transversal to Σ_0', there is a unique germ at 0 of a holomorphic mapping $H\colon(\mathbb{C}^N,0)\to(\mathbb{C}^{N'},0)$ satisfying (1.7).

By combining Theorem 1.6 with Theorem 3.1 in ref. [7], we obtain the following result concerning the biholomorphic equivalence problem.

Corollary 1.7. Let M and d be as in Theorem 1.1. Then there exist an open set Ω and $A_1,\dots,A_d,B_1,\dots,B_{d+1}$ as in Theorem 1.6 such that the following holds. If M' is a real-analytic generic submanifold of codimension d through 0 in $\mathbb{C}^N$, then there exists a germ at 0 of a holomorphic mapping $\Psi\colon((\Sigma_0')^{2(d+1)},0)\to(\mathbb{C}^d,0)$ such that M and M' are biholomorphically equivalent at 0 if and only if there exists a germ at 0 of a holomorphic mapping $F\colon(\mathbb{C}^N,0)\to(\Sigma_0',0)$ such that $F|_{\Sigma_0}$ is a local biholomorphism at 0 and (1.8) holds. Here Σ_0 denotes the Segre variety of M at 0.

We would like to point out that the hypothesis of finite type in Theorems 1.1 and 1.6 is crucial as is illustrated by the following simple example.

Example 1.8. Let $M\subset\mathbb{C}^2$ be the real-analytic hypersurface given by

$$\operatorname{Im} w = (\operatorname{Re} w)|z|^2,$$

which is of finite type at all points except along $\{w=0\}$. Note that the family of holomorphic mappings

$$H_t(z,w)=(z,tw),$$

for all $t\in\mathbb{R}$, sends M into itself. Thus, the conclusion of Theorem 1.1 (with $\tilde{z}(z,w)=(z,0)\in\Sigma_0$) does not hold. Also, for any holomorphic function $F(z,w)$, with $F(0)=0$, the mapping

$$H(z,w)=(F(z,w),0)$$

sends M into itself and, hence, in contrast with the conclusion of Theorem 1.6, there is no (non-trivial) condition on a mapping F to be a component of a holomorphic mapping M into itself.

As an application of Theorem 1.1, we give a refinement of some results concerning finite jet determination of holomorphic mappings between generic submanifolds (see sec. 5 for details). This is a problem that has received much attention recently. We mention here the papers[8–13], where results on finite jet determination of mappings between generic submanifolds are obtained. The reader is also referred to the survey papers [14,15] for further references and results.

2 An invariant description of normal coordinates

For the proof of Theorem 1.1, we shall need the following description of all normal coordinates for a real-analytic generic submanifold. Let $M\subset\mathbb{C}^N$ be a real-analytic generic submanifold through 0 of codimension d. Recall that local holomorphic coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$, with $n = N - d$, are called normal if M can be expressed near 0 as a graph of the form

$$\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w), \tag{2.1}$$

where $\phi(z, \bar{z}, s)$ is an $\mathbb{R}^d$-valued real-analytic function in a neighborhood of 0 in $\mathbb{C}^n \times \mathbb{R}^d$ with

$$\phi(z, 0, s) \equiv \phi(0, \chi, s) \equiv 0. \tag{2.2}$$

Equivalently, M can be defined by a complex equation of the form

$$w = Q(z, \bar{z}, \bar{w}), \tag{2.3}$$

where $Q(z, \chi, \tau)$ is a $\mathbb{C}^d$-valued holomorphic function, defined near 0 in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d$, satisfying

$$Q(z, 0, \tau) \equiv Q(0, \chi, \tau) \equiv \tau. \tag{2.4}$$

Normal coordinates were first introduced by Chern-Moser[16] (see also ref. [17]).

As in the beginning of sec. 1, let U be a small open neighborhood of 0 in $\mathbb{C}^N$, and Σ_0 the Segre variety of M at 0. Let $\tilde{z}: U \to \Sigma_0$ be a holomorphic submersion such that $\tilde{z}(0) = 0$ and the d-dimensional complex submanifold $W := \tilde{z}^{-1}(0)$ is transversal to Σ_0 at 0. Observe that, for $p \in \mathbb{C}^N$ sufficiently close to 0, the submanifolds Σ_p and W also intersect transversally near 0. Hence, after shrinking U if necessary, we may define a mapping $\sigma: U \to W$ by letting $\sigma(p)$ be the single point of intersection between Σ_p and W. We denote by $\iota: W \to W$ the restriction of σ to W, i.e.

$$\Sigma_p \cap W = \{\iota(p)\}, \quad p \in W. \tag{2.5}$$

It follows that, for $p \in W$ sufficiently close to 0, $\iota^2(p) = p$ since $\iota(p) \in \Sigma_p \cap W$ and hence, ι is a local involution on W.

Theorem 2.1. Let M be a real-analytic generic submanifold of codimension d in $\mathbb{C}^N$ with $0 \in M$, and Σ_p its Segre variety at p, for p close to 0. Let $U \subset \mathbb{C}^N$ be a sufficiently small neighborhood of 0 and $\tilde{z}: (U, 0) \to (\Sigma_0, 0)$ be a holomorphic submersion such that the d-dimensional complex submanifold $W := \tilde{z}^{-1}(0)$ is transversal to Σ_0 at 0, and $\iota: W \to W$ the corresponding mapping as defined in (2.5). Then there are open neighborhoods of the origin $V \subset \mathbb{C}^N$, $X \subset \Sigma_0$, $Y \subset W$ such that the following hold:

(i) $\iota: Y \to Y$ is an anti-holomorphic involution fixing $M \cap Y$.

(ii) There is a unique holomorphic submersion $\tilde{w}: (V, 0) \to (Y, 0)$ such that the mapping $H: (V, 0) \to (X \times Y, 0)$, where $H(Z) = (\tilde{z}(Z), \tilde{w}(Z))$, is a biholomorphism satisfying

$$H(\Sigma_p \cap V) = X \times \{\iota(p)\}, \quad \forall p \in Y. \tag{2.6}$$

(iii) If $\alpha: (X, 0) \to (\mathbb{C}^n, 0)$, and $\beta: (Y, 0) \to (\mathbb{C}^d, 0)$ are biholomorphisms, then in the coordinates $(z, w) := (\alpha \circ \tilde{z}, \beta \circ \tilde{w})$ in $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^d$ the submanifold M is given near 0 by $w = Q(z, \bar{z}, \bar{w})$, where $Q(z, \chi, \tau)$ is a $\mathbb{C}^d$-valued holomorphic function satisfying

$$Q(z, 0, \tau) \equiv Q(0, \chi, \tau) \equiv \hat{\iota}(\bar{\tau}) \tag{2.7}$$

with $\hat{\iota}$ the involution given by $\beta \circ \iota \circ \beta^{-1}$. Moreover, if $\beta(Y \cap M) \subset \mathbb{R}^d$, then $\hat{\iota}(w) = \bar{w}$ and, hence, (z, w) are normal coordinates, i.e. the identity (2.4) holds.

Proof of Theorem 2.1. We let $\rho(Z, \bar{Z})$ be a defining function for M as in the introduction. Consider the germ at 0 of a holomorphic mapping $f: (\mathbb{C}^N \times \mathbb{C}^N, 0) \to (\mathbb{C}^d \times \Sigma_0, 0)$ defined by

$$f(Z, \zeta) := (\rho(Z, \zeta), \widetilde{z}(Z)), \tag{2.8}$$

and the equation

$$f(Z, \zeta) = (0, t). \tag{2.9}$$

We claim that $Z \mapsto f(Z, 0)$ is a local biholomorphism at 0. Indeed,

$$\frac{\partial f}{\partial Z}(0,0) = \left(\frac{\partial \rho}{\partial Z}(0,0), \frac{\partial \widetilde{z}}{\partial Z}(0)\right) \tag{2.10}$$

and, hence, the claim follows from the transversality of the intersection between Σ_0 and W at 0. By the implicit function theorem, there exists a unique germ at 0 of a holomorphic mapping $Z = \gamma(t, \zeta)$ from $(\Sigma_0 \times \mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ that solves equation (2.9). It follows from (2.9) that $t \mapsto \gamma(t, 0)$ has rank $n := N - d$ at 0 and that $t \mapsto \gamma(t, \bar{p})$, for $p \in \mathbb{C}^N$ sufficiently close to 0, parametrizes an open piece of the Segre variety Σ_p. We observe, from the definition of γ, that $\sigma(p) = \gamma(0, \bar{p})$, for $p \in \mathbb{C}^N$ close to 0; recall that $\sigma(p)$ denotes the single point of intersection between Σ_p and W, as defined above. In particular,

$$\iota(p) = \gamma(0, \bar{p}), \tag{2.11}$$

where ι is the involution of W defined by(2.5), and hence the anti-holomorphic mapping $p \mapsto \gamma(0, \bar{p})$ is a local diffeomorphism at 0 of W. It follows that the mapping $(t, p) \mapsto \gamma(t, \bar{p})$ from $\Sigma_0 \times W \to \mathbb{C}^N$ is holomorphic in t, anti-holomorphic in p, and is a local diffeomorphism at 0. Hence, if we denote by W^* the submanifold $\{Z: \bar{Z} \in W\}$, then the mapping $\Gamma(t, p) := \gamma(t, p)$ from $\Sigma_0 \times W^* \to \mathbb{C}^N$ is a local biholomorphism at 0. As a consequence, we may define the germ at 0 of a holomorphic mapping $\widetilde{w}: (\mathbb{C}^N, 0) \to (W, 0)$ by

$$\widetilde{w}(\Gamma(t, p)) = \gamma(0, p) \ (= \iota(\bar{p})). \tag{2.12}$$

Since $p \mapsto \iota(p)$ is a local diffeomorphism of W at 0, it follows that $\widetilde{w}$ is a submersion at 0. Since ι is a local involution on W, we can find a sufficiently small open neighborhood Y of 0 in W such that ι is an involution on Y (i.e. ι maps Y onto itself and ι^2 is the identity). Indeed, if Y_0 is any sufficiently small open neighborhood of 0 in W, then $Y: = Y_0 \cap \iota(Y_0)$ is such a neighborhood. This proves (i). To prove (ii), let X be any sufficiently small open neighborhood of 0 in Σ_0 and define $V: = \Gamma(X \times Y^*)$. Then $\widetilde{w}$, defined by (2.12) is a holomorphic submersion $(V, 0) \to (Y, 0)$. Observe that, for every $p \in Y$, $X \ni t \mapsto \Gamma(t, \bar{p}) \in V$ parametrizes $\Sigma_p \cap V$. Hence, eq. (2.6) in Theorem 2.1 is equivalent to (2.12). This proves (ii).

To prove (iii), we assume that α and β are as in (iii), and let (z, w) be the coordinates $(z, w) := (\alpha \circ \widetilde{z}, \beta \circ \widetilde{w})$. In these coordinates, it follows from (ii) that $\Sigma_0 = \{(z, w): w = 0\}$. Consequently, if $\hat{\rho}(z, w, \bar{z}, \bar{w}) = 0$ is a defining equation for M in the coordinates

(z,w), then $\det \partial\hat{\rho}/\partial w(0) \neq 0$. Hence, by the implicit function theorem, we may solve for w in the equation $\hat{\rho}(z,w,\bar{z},\bar{w}) = 0$ and obtain a defining equation for M of the form (2.3). The fact that

$$Q(z,0,\tau) = \hat{\iota}(\bar{\tau}) \tag{2.13}$$

is a direct consequence of (2.6). To prove the remaining part of (2.7), we note, by the fact that (2.3) defines a real submanifold, that we have

$$Q(z,\chi,\bar{Q}(\chi,z,w)) \equiv w. \tag{2.14}$$

By substituting $z = 0$ in (2.14) and using (2.13), we obtain $Q(0,\chi,\overline{\hat{\iota}(w)}) = w$. The desired identity $Q(0,\chi,\tau) = \hat{\iota}(\bar{\tau})$ follows by taking $w = \hat{\iota}(\bar{\tau})$.

If $\beta(Y \cap M) \subset \mathbb{R}^d$, then, since $Y \cap M$ is the fixed point set of the involution ι, it follows that $\mathbb{R}^d \cap \beta(Y)$ is the fixed point set of the anti-holomorphic involution $\hat{\iota}$ on $\beta(Y) \subset \mathbb{C}^d$, i.e. $\hat{\iota}(w) = \bar{w}$. The identity (2.4) follows immediately. The fact that M can be graphed as in (2.1) with ϕ satisfying (2.2) is a direct consequence of the implicit function theorem and (2.4). This completes the proof of Theorem 2.1.

Remark 2.2. It is not difficult to see that all normal coordinates (z,w) are obtained in the way described by Theorem 2.1 for some choice of submersion $\tilde{z}$. The details of this are left to the reader.

Remark 2.3 Let $Z = (\hat{z},\hat{w})$ be given coordinates in $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^d$ in which the Segre variety Σ_0 of M at 0 is tangent to $\{(\hat{z},\hat{w})\colon \hat{w} = 0\}$ at 0. As a consequence of Theorem 2.1 (and its proof), we obtain the following description of all possible holomorphic transformations $(z,w) = (F(\hat{z},\hat{w}),G(\hat{z},\hat{w}))$ yielding normal coordinates for M. Let $F\colon (\mathbb{C}^N,0) \to (\mathbb{C}^n,0)$ be an arbitrary local holomorphic mapping with $\det(\partial F/\partial\hat{z})(0) \neq 0$. In the setting of the theorem, this corresponds to a choice of a holomorphic submersion $\tilde{z}$ and a local chart α of Σ_0, with $F = \alpha \circ \tilde{z}$. We note that $\tilde{z}$ and α are not uniquely determined by F. However, any two different choices of $\tilde{z}$ differ only by a reparametrization of Σ_0. An inspection of the proof of Theorem 2.1 shows that the $\tilde{w}$ given by (ii) in the theorem is uniquely determined by F. Moreover, for any local chart β on W such that $\beta(M \cap W) \subset \mathbb{R}^d$, the mapping $G = \beta \circ \tilde{w}$ produces normal coordinates by (iii) of Theorem 2.1. It is easily seen that if we write $G(0,\hat{w}) = g_1(\hat{w}) + ig_2(\hat{w})$, where g_1 and g_2 are real-valued on $\mathbb{R}^d$, then there is a one-to-one correspondence between choices of such parametrizations β and choices of $g_1(\hat{w})$ with $\det(\partial g_1/\partial\hat{w})(0) \neq 0$. We conclude that $G(\hat{z},\hat{w})$ is uniquely determined by $F(\hat{z},\hat{w})$ and an arbitrary choice of $g_1(\hat{w})$ with $\det(\partial g_1/\partial\hat{w})(0) \neq 0$.

3 Proof of Theorem 1.6 in the case of hypersurfaces in $\mathbb{C}^2$

In order to illustrate the idea of the proof of Theorem 1.6, we first give a proof for the case of hypersurfaces M, M' in $\mathbb{C}^2$. Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface of finite type at $0 \in M$. Assume that $(z,w) \in \mathbb{C} \times \mathbb{C}$ are normal coordinates at 0. Thus M is given locally near 0 by (2.3), where the scalar-valued holomorphic function $Q(z,\chi,\tau)$ satisfies (2.4). The finite type condition on the hypersurface M is equivalent to $Q(z,\chi,0) \not\equiv 0$, which implies, by the normality of (z,w) that

$$Q_\chi(z,\chi,0) \not\equiv 0, \tag{3.1}$$

where we use the notation $Q_\chi = \partial Q/\partial\chi$. The first four iterated Segre mappings (as defined in refs. [2, 18] are given by:

$$\begin{aligned} &v^1(t^1) := (t^1,0), \quad v^2(t^1,t^2) := (t^2, Q(t^2,t^1,0)),\\ &v^3(t^1,t^2,t^3) := (t^3, Q(t^3,t^2,\bar{Q}(t^2,t^1,0))),\\ &v^4(t^1,t^2,t^3,t^4) := (t^4, Q(t^4,t^3,\bar{Q}(t^3,t^2,Q(t^2,t^1,0)))). \end{aligned} \tag{3.2}$$

For convenience, we shall also write $v^k(t^1,\dots,t^k) = (t^k, u^k(t^1,\dots,t^k))$.

We let $M' \subset \mathbb{C}^2$ be another real-analytic hypersurface through 0 and $(\widetilde{z},\widetilde{w})$ normal coordinates for M'. We refer to the corresponding objects for M' by the addition of $\widetilde{\ }$. Let $H = (F,G)$ be a germ at 0 of a holomorphic mapping with $H(0) = 0$. If H sends M into M', then (see ref. [7], sec. 2)

$$G\circ v^4 = \widetilde{u}^4(\overline{F\circ v^1}, F\circ v^2, \overline{F\circ v^3}, F\circ v^4). \tag{3.3}$$

Conversely, if H satisfies (3.3), then we claim that H sends M into M'. Indeed, if we take $t^1 = 0$ in (3.3) and complex conjugate, then we obtain

$$\overline{G\circ v^3} = \overline{\widetilde{u}^3}(F\circ v^1, \overline{F\circ v^2}, F\circ v^3) \tag{3.4}$$

by using standard properties of the iterated Segre mappings (see ref. [18] or ref. [2]). We now observe that

$$\widetilde{u}^4(\bar{t}^1,\bar{t}^2,\bar{t}^3,\bar{t}^4) = \widetilde{Q}(\bar{t}^4,\bar{t}^3,\overline{\widetilde{u}^3}(\bar{t}^1,\bar{t}^2,\bar{t}^3)). \tag{3.5}$$

By using (3.5) and (3.4) in (3.3), we conclude that

$$G\circ v^4 = \widetilde{Q}(F\circ v^4, \overline{F\circ v^3}, \overline{G\circ v^3}). \tag{3.6}$$

Let $\mathcal{M} \subset \mathbb{C}^2\times\mathbb{C}^2$ be the complexification of M, i.e. the complex submanifold through 0 in $\mathbb{C}^2\times\mathbb{C}^2$ defined by $w = Q(z,\chi,\tau)$. Since $(t^1,t^2,t^3,t^4) \mapsto (v^4(t^1,t^2,t^3,t^4), \overline{v^3}(t^1,t^2,t^3))$ is a holomorphic mapping of generic full rank into $\mathcal{M}$ (see ref. [18] or ref. [2]), we conclude that $G(z,w) = \widetilde{Q}(F(z,w),\bar{F}(\chi,\tau),\bar{G}(\chi,\tau))$ for all (z,w,χ,τ) on $\mathcal{M}$ and, hence, H sends M into M'. This proves the claim.

Consider the equation $(z,w) = v^4(t^1,t^2,t^3,t^4)$, which can also be written in the form $z = t^4$ and

$$w = Q(z,t^3,\bar{Q}(t^3,t^2,Q(t^2,t^1,0))). \tag{3.7}$$

We make the linear change of variables

$$\eta^1 = \frac{t^1+t^3}{2}, \quad \eta^2 = t^2, \quad \sigma = \frac{t^1-t^3}{2} \tag{3.8}$$

and obtain

$$w = Q(z,\eta^1-\sigma,\bar{Q}(\eta^1-\sigma,\eta^2,Q(\eta^2,\eta^1+\sigma,0))). \tag{3.9}$$

Let us use the notation $\eta = (\eta^1,\eta^2)$ and write

$$U(\eta,z,\sigma) := Q(z,\eta^1-\sigma,\bar{Q}(\eta^1-\sigma,\eta^2,Q(\eta^2,\eta^1+\sigma,0))).$$

We have $U(\eta,0,0)\equiv 0$ and

$$\begin{aligned}\Delta(\eta) &:= \frac{\partial}{\partial\sigma}U(\eta,z,\sigma)\big|_{z=\sigma=0}\\ &= -\bar{Q}_\chi(\eta^1,\eta^2,Q(\eta^2,\eta^1,0))+\bar{Q}_w(\eta^1,\eta^2,Q(\eta^2,\eta^1,0))Q_\chi(\eta^2,\eta^1,0). \qquad (3.10)\end{aligned}$$

Here we have used the notation $\bar{Q}(\chi,z,w)$ and hence the corresponding derivatives $\bar{Q}_\chi$ and $\bar{Q}_w$ refer to partial derivatives with respect to the first and last variable, respectively. By differentiating the identity

$$\bar{Q}(\eta^1,\eta^2,Q(\eta^2,\eta^1,0))\equiv 0$$

with respect to η^1, we obtain from (3.10)

$$\Delta(\eta)=2\bar{Q}_w(\eta^1,\eta^2,Q(\eta^2,\eta^1,0))Q_\chi(\eta^2,\eta^1,0)\not\equiv 0,$$

in view of (3.1) and the fact that $\bar{Q}_w(0,0,0)=1$. We may now apply the singular implicit function theorem given in Proposition 4.1.18 in ref. [18] and conclude that eq. (3.9) has a unique solution of the form

$$\sigma=\Theta\Big(\eta,\frac{z}{\Delta(\eta)^2},\frac{w}{\Delta(\eta)^2}\Big), \qquad (3.11)$$

where $\Theta(\eta,z',w')$ is holomorphic near $0\in\mathbb{C}^4$ and $\Theta(\eta,0,0)\equiv 0$. If we now substitute for (t^1,t^2,t^3,t^4) in (3.3) using the linear change of variables (3.8), $t^4=z$, and then substitute for σ using (3.11), then we obtain

$$\begin{aligned}G(z,w)=\tilde{u}^4(&\overline{F\circ v^1}(\eta^1+\Theta),F\circ v^2(\eta^1+\Theta,\eta^2),\\ &\overline{F\circ v^3}(\eta^1+\Theta,\eta^2,\eta^1-\Theta),F(z,w)), \qquad (3.12)\end{aligned}$$

where Θ is given by the right hand side of (3.11). In particular, if H sends M into M', then the right hand side of (3.12) is independent of the variable η. Conversely, if $F(z,w)$ is such that the right hand side of (3.12) is independent of the variable η, then we can define $G(z,w)$ by (3.12). We claim that $H=(F,G)$ sends M into M'. Indeed, for any $\eta\in\mathbb{C}^2$ sufficiently close to 0 with $\Delta(\eta)\neq 0$, we have

$$\Theta\Big(\eta,\frac{z}{\Delta(\eta)^2},\frac{U(\eta,z,\sigma)}{\Delta(\eta)^2}\Big)=\sigma, \qquad (3.13)$$

for all sufficiently small z and σ, by the uniqueness of the solution (3.11) to eq. (3.9). We now make the subsitution $(z,w)=v^4(t^1,t^2,t^3,t^4)$, $\eta^1=(t^1+t^3)/2$, $\eta^2=t^2$ in (3.12). Using again the linear change of variables (3.8), and $t^4=z$ in the identity (3.13), we conclude, since in these variables we have $u^4(t^1,t^2,t^3,t^4)=U(\eta,z,\sigma)$, that (3.3) holds. This proves the claim, in view of the remarks above.

This proves Theorem 1.6 for hypersurfaces in $\mathbb{C}^2$ with

$$\begin{aligned}&\Psi(\overline{F}\circ B_1(Z,\eta),F\circ A_1(Z,\eta),\overline{F}\circ B_2(Z,\eta),F(Z))\\ &=\tilde{u}^4(\overline{F\circ v^1}(\eta^1+\Theta),F\circ v^2(\eta^1+\Theta,\eta^2),\overline{F\circ v^3}(\eta^1+\Theta,\eta^2,\eta^1-\Theta),F(z,w)), \qquad (3.14)\end{aligned}$$

and hence

$$\begin{aligned}A_1(z,w,\eta)&=v^2(\eta^1+\Theta,\eta^2),\\ B_1(z,w,\eta)&=\overline{v^1}(\eta^1+\Theta),\\ B_2(z,w,\eta)&=\overline{v^3}(\eta^1+\Theta,\eta^2,\eta^1-\Theta),\end{aligned}$$

where Θ is given by the right hand side of (3.11). The open set $\Omega \subset \mathbb{C}^2 \times \mathbb{C}^2$ can be taken to be of the form

$$|\eta| < \epsilon, \quad \Delta(\eta) \neq 0, \quad |z| < \epsilon\Delta(\eta), \quad |w| < \epsilon\Delta(\eta),$$

for $\epsilon > 0$ sufficiently small.

4 Proof of Theorem 1.6 in the general case and proof of Theorem 1.1

Proof of Theorem 1.6. We point out that, in view of Theorem 2.1, it suffices to prove Theorem 1.6 in some sets of normal coordinates for M and M'. The proof in the general case parallels that for hypersurfaces in $\mathbb{C}^2$ given in sec. 3. Let $M \subset \mathbb{C}^N$ be given in normal coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ near the origin in $\mathbb{C}^N$, i.e. by (2.3) where $Q(z, \chi, \tau)$ is a $\mathbb{C}^d$-valued holomorphic function satisfying (2.4). The iterated Segre mappings (see ref. [2, 18]) $v^j: (\mathbb{C}^{jn}, 0) \to (\mathbb{C}^N, 0)$ are given in these coordinates by $v^1(t^1) = (t^1, 0)$ and, recursively, for $j \geqslant 1$ by

$$v^{j+1}(t^1, \dots, t^{j+1}) = (t^{j+1}, u^{j+1}(t^1, \dots, t^{j+1})) = (t^{j+1}, Q(t^{j+1}, \overline{v^j}(t^1, \dots, t^j))). \quad (4.1)$$

We let $M' \subset \mathbb{C}^{N'}$ be another real-analytic generic submanifold through 0 and $(\widetilde{z}, \widetilde{w})$ normal coordinates for M'. We refer to the corresponding objects for M' by the addition of $\widetilde{\ }$. Let $H = (F, G)$ be a germ at 0 of a holomorphic mapping $(\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ with $H(0) = 0$. Let $m := d + 1$. As in sec. 3, if H sends M into M' (cf. ref. [7], sec. [2]), then

$$G \circ v^{2m} = \widetilde{u}^{2m}(\overline{F \circ v^1}, F \circ v^2, \dots, \overline{F \circ v^{2m-1}}, F \circ v^{2m}). \quad (4.2)$$

Conversely, if (4.2) holds, then, since the mapping

$$(t^1, \dots, t^{2m}) \mapsto (v^{2m}(t^1, \dots, t^{2m}), \overline{v^{2m-1}}(t^1, \dots, t^{2m-1}))$$

has generic full rank as a holomorphic mapping into the complexification $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ (see ref. [18] or ref. [2]), a similar argument to that in sec. 3 shows that H sends M into M'.

As in sec. 3, we consider the equation $(z, w) = v^{2m}(t^1, \dots, t^{2m})$ or, equivalently, $z = t^{2m}$ and

$$w = u^{2m}(t^1, \dots, t^{2m-1}, z). \quad (4.3)$$

We make the linear change of variables

$$\eta^j = \frac{t^j + t^{2m-j}}{2}, \quad \sigma^j = \frac{t^j - t^{2m-j}}{2}, \quad j = 1, \dots, m-1, \quad (4.4)$$

and

$$\eta^m = t^m. \quad (4.5)$$

Thus, if we write $\eta = (\eta^1, \dots, \eta^m)$, $\sigma = (\sigma^1, \dots, \sigma^m)$, and

$$U(\eta, z, \sigma) := u^{2m}(\eta^1 + \sigma_1, \dots, \eta^{m-1} + \sigma_{m-1}, \eta^m, \eta^1 - \sigma_1, \dots, \eta^{m-1} - \sigma_{m-1}, z) \quad (4.6)$$

then eq. (4.3) becomes

$$w = U(\eta, z, \sigma). \quad (4.7)$$

Since M is of finite type at 0, it follows from Lemma 4.1.3 in [18] that $U(\eta,0,0)\equiv 0$ and that we may decompose $\sigma=(\sigma',\sigma'')\in\mathbb{C}^d\times\mathbb{C}^{(m-1)n-d}$, after reordering the variables if necessary, such that

$$\Delta(\eta):=\det\Big(\frac{\partial}{\partial\sigma'}U(\eta,z,\sigma',\sigma'')\big|_{z=0,\sigma=0}\Big)\not\equiv 0. \tag{4.8}$$

(The reader should be warned that the notation in ref. [18] is slightly different from that of the present paper.) We should point out that in the hypersurface case in sec. 3 these facts were easily verified directly by using the condition of finite type. In the case of higher codimension, the proof of these facts is more involved. We may now apply the singular implicit function theorem given in Proposition 4.1.18 of ref. [18] and solve for σ' in eq. (4.3) and obtain a unique solution of the form

$$\sigma'=\Theta\Big(\eta,\frac{\sigma''}{\Delta(\eta)^2},\frac{z}{\Delta(\eta)^2},\frac{w}{\Delta(\eta)^2}\Big), \tag{4.9}$$

where $\Theta(\eta,\sigma'',z,w)$ is a $\mathbb{C}^d$-valued holomorphic function near 0 in $\mathbb{C}^{mn}\times\mathbb{C}^{(m-1)n-d}\times\mathbb{C}^n\times\mathbb{C}^d$ with $\Theta(\eta,0,0,0)=0$. Substituting for $(t^1,\ldots,t^{2m})$ in terms of η, σ, and z in (4.2) using (4.4), $t^m=\eta^m$, $t^{2m}=z$, and then substituting for σ' using (4.9), we obtain a relation of the form (cf. (3.12))

$$\begin{aligned}G(z,w)=\Psi(&\overline{F}\circ B_1(z,w,\xi),F\circ A_1(z,w,\xi),\ldots,\\&\overline{F}\circ B_d(z,w,\xi),F\circ A_d(z,w,\xi),\overline{F}\circ B_{d+1}(z,w,\xi),F(z,w)),\end{aligned} \tag{4.10}$$

where $\xi=(\eta,\sigma'')\in\mathbb{C}^{nm}\times\mathbb{C}^{(m-1)n-d}$. The A_i and B_j are obtained by making the substitutions described above for $t^1,\ldots,t^{2m}$ in the iterated Segre mappings $v^1,\ldots,v^{2m}$ and their complex conjugates. The reader can verify from the construction of Θ that $A_i(0,\xi)$ and $B_j(0,\xi)$ tend to 0 as $\xi\to 0$. If $H=(F,G)$ sends M into M', then the right hand side of (4.10) is independent of ξ or, equivalently, (1.8) holds. The converse follows in the same way as in the hypersurface case in sec. 3. This completes the proof of Theorem 1.6.

Proof of Theorem 1.1. We keep the notation of the proof of Theorem 1.6. First, by using Theorem 2.1, we can find normal coordinates $(\widetilde{z},\widetilde{w})$ for M' such that $\widetilde{z}=\widetilde{z}(\widetilde{z})$ is the submersion given in Theorem 1.1. If H is a germ at 0 of a holomorphic mapping $(\mathbb{C}^N,0)\to(\mathbb{C}^{N'},0)$, then, in these coordinates, $H=(F,G)$ with $F=\widetilde{z}\circ H$. Hence, to prove Theorem 1.1, it suffices to show that we have an identity of the form

$$G=\Psi\circ(\overline{F}\circ h_1^\lambda,F\circ g_1^\lambda,\ldots,\overline{F}\circ h_d^\lambda,F\circ g_d^\lambda,\overline{F}\circ h_{d+1}^\lambda,F), \tag{4.11}$$

for all λ sufficiently small. Let $D\colon\mathbb{D}\to\mathbb{C}^{nm}$ be a holomorphic mapping with $D(0)=0$ such that $\Delta(D(\lambda))\neq 0$ for $\lambda\neq 0$, where $\Delta(\eta)$ is the determinant given by (4.8). The conclusion of Theorem 1.1 now follows by substituting $\xi=(\eta,\sigma'')=(D(\lambda),0)$ in the identity (4.10).

Remark 4.1. In the case where M is a Levi-nondegenerate hypersurface it is possible to prove a version of Theorem 1.6 with fewer parameters by using the iterated Segre map v^3 rather than v^4. We shall illustrate this in the model case where $M\subset\mathbb{C}^{n+1}$

and $M' \subset \mathbb{C}^{n'+1}$ are respectively given by

$$\operatorname{Im} w = \langle z, \bar{z} \rangle_n = \sum_{j=1}^{n} |z_j|^2 \quad \text{and} \quad \operatorname{Im} \widetilde{w} = \langle \widetilde{z}, \bar{\widetilde{z}} \rangle_{n'} = \sum_{j=1}^{n'} |\widetilde{z}_j|^2.$$

By a calculation similar to that given in sec. 3, one obtains that a germ at 0 of a mapping $H = (F, G) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n'+1}, 0)$ sends M into M' if and only if

$$G(z,w) = 2i\Big\langle F(z,w) - F\Big(\frac{\langle z, t^2\rangle_n - \langle t^1_*, t^2_* \rangle_{n-1} + iw/2}{t^2_1}, t^1_*, 0\Big), \overline{F}(t^2, w - 2i\langle z, t^2 \rangle_n)\Big\rangle_{n'}, \tag{4.12}$$

where we have used the notation $t^j = (t^j_1, t^j_*) \in \mathbb{C} \times \mathbb{C}^{n-1}$, for $j = 1, 2$. Here equation (1.8) is equivalent to the condition that the right hand side of (4.12) is independent of the $2n - 1$ parameters t^1_* and t^2. Note that Theorem 1.6 involves $3n - 1$ parameters in the case that M is a hypersurface.

Remark 4.2. We point out that the independence of the right hand side of (4.11) on the parameter λ is not sufficient to guarantee that F is a component of a mapping sending M into M'. This is the case even in the context of self mappings of the Levi-nondegenerate hypersurface M given in Remark 4.1 above with $n > 1$.

5 An application to the problem of finite jet determination

Let $\mathcal{F}$ be a class of germs of holomorphic mappings $(X, x) \to (Y, y)$, where X and Y are complex manifolds with $x \in X$ and $y \in Y$, respectively. We shall say that $\mathcal{F}$ satisfies the finite jet determination property at x if there exists an integer $k \geqslant 0$ such that for any pair $H^1, H^2 \in \mathcal{F}$, the condition $j^k_x H = j^k_x H'$ implies $H \equiv H'$. Here, $j^k_x H$ denotes the k-jet at x of H. For instance, if M and M' are real-analytic generic submanifolds of codimension d through 0 in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ respectively and $\mathcal{F}$ is a class of germs at 0 of holomorphic mappings $(\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ sending M into M', then there are a number of sufficient conditions that can be imposed on M (or M') to guarantee that $\mathcal{F}$ satisfies the finite jet determination property (see refs. [8–13]). As a consequence of Theorem 1.1, we obtain the following result.

Theorem 5.1. Let M and M' be real-analytic generic submanifolds through 0 in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively, with M of finite type at 0. Let $\mathcal{F}$ be a class of germs at 0 of holomorphic mappings $(\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ sending M into M' such that $\mathcal{F}$ satisfies the finite jet determination property. Then there exists an integer $k > 0$ with the following property. Let $\widetilde{z}$ be a germ at 0 of a holomorphic submersion $\widetilde{z}\colon (\mathbb{C}^{N'}, 0) \to (\Sigma'_0, 0)$, where Σ'_0 is the Segre variety of M' at 0, such that $\widetilde{z}^{-1}(0)$ is transversal to Σ'_0. If $H^1, H^2 \in \mathcal{F}$ and $j^k_0(\widetilde{z} \circ H^1) = j^k_0(\widetilde{z} \circ H^2)$, then $H^1 \equiv H^2$.

By using a result from ref. [10], we immediately obtain the following corollary. Recall that a generic submanifold is called holomorphically nondegenerate at 0 if there are no germs at $0 \in M$ of (non-trivial) holomorphic vector fields (i.e. $(1, 0)$ vector fields with holomorphic coefficients) that are tangent to M in a neighborhood of 0.

Corollary 5.2. Let M and M' be real-analytic generic submanifolds of codimension d through 0 in $\mathbb{C}^N$ with M of finite type and holomorphically nondegenerate at 0. Let $(\tilde{z},\tilde{w}) \in \mathbb{C}^{N-d} \times \mathbb{C}^d$ be normal coordinates for M' at 0. Then there exists an integer $k \geqslant 0$ with the following property. Let $H^1, H^2: (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$ be germs at 0 of local biholomorphisms sending M into M' and $H^j = (F^j, G^j)$, $j = 1,2$, in the coordinates $(\tilde{z},\tilde{w})$. If $j_0^k F^1 = j_0^k F^2$, then $H^1 \equiv H^2$.

Recall that a germ at 0 of a real-analytic hypersurface $M \subset \mathbb{C}^N$ is of D'Angelo finite type[19] if there is no germ of a nontrivial complex curve C through 0 contained in M. By using a recent result of Lamel-Mir[20] on finite jet determination for all mappings between hypersurfaces of D'Angelo finite type, we obtain the following .

Corollary 5.3. Let M and M' be real-analytic hyperfaces in $\mathbb{C}^{n+1}$ of D'Angelo finite type at 0, with $(\tilde{z},\tilde{w}) \in \mathbb{C}^n \times \mathbb{C}$ normal coordinates for M' at 0. Then there exists an integer $k > 0$ with the following property. Let $H^1, H^2: (\mathbb{C}^{n+1},0) \to (\mathbb{C}^{n+1},0)$ be germs at 0 of holomorphic mappings sending M into M' and $H^j = (F^j, G^j)$, $j = 1,2$, in the coordinates $(\tilde{z},\tilde{w})$. If $j_0^k F^1 = j_0^k F^2$, then $H^1 \equiv H^2$.

Proof of Theorem 5.1. Assume that the mappings in the class $\mathcal{F}$ are determined by their k_0-jets at 0. In view of Theorem 2.1, it suffices to take normal coordinates $(\tilde{z},\tilde{w}) \in \mathbb{C}^{n'} \times \mathbb{C}^{d'}$ for M' at 0, write $H = (F,G)$ in these coordinates, and show that there exists a $k \geqslant k_0$ such that the k_0-jet of G at 0 is determined by the k-jet of F at 0. We start with eq. (4.11), which in view of Remark (1.2) can be written as follows

$$G(Z) = \Psi\Big(\overline{F}\Big(\hat{h}_1\Big(\frac{Z}{\lambda^l},\lambda\Big)\Big), F\Big(\hat{g}_1\Big(\frac{Z}{\lambda^l},\lambda\Big)\Big), \dots,$$
$$\overline{F}\Big(\hat{h}_d\Big(\frac{Z}{\lambda^l},\lambda\Big)\Big), F\Big(\hat{g}_d\Big(\frac{Z}{\lambda^l},\lambda\Big)\Big), \overline{F}\Big(\hat{h}_{d+1}\Big(\frac{Z}{\lambda^l},\lambda\Big)\Big), F(Z)\Big), \tag{5.1}$$

where the $\hat{h}_j$ and $\hat{g}_j$ are as in (1.6). By differentiating (5.1) repeatedly with respect to Z, setting $Z = 0$, we conclude that

$$\frac{\partial^{|\alpha|}G}{\partial Z^\alpha}(0) = \sum_{j=-|\alpha|l}^{\infty} a_j^\alpha \lambda^j, \tag{5.2}$$

where each coefficient a_j^α is a polynomial in the components of $j_0^{|\alpha|(l+1)+j}F$ for all $j \geqslant -|\alpha|l$. Since the left hand side of (5.1) (and hence of (5.2)) is independent of λ, the coefficient $a_j^\alpha = 0$ for $j \neq 0$ and $(\partial^{|\alpha|}G/\partial Z^\alpha)(0) = a_0^\alpha$. This completes the proof with $k = k_0(l+1)$.

Remark 5.4. The proof of Theorem 5.1 shows that if the mappings H in $\mathcal{F}$ are determined by their k_0-jets at 0, then they are also determined by the k-jets at 0 of $\tilde{z} \circ H$, where $k = k_0(l+1)$ and l is the integer given in Remark 1.2 (depending only on M). However, we do not know any example where the k_0-jet at 0 of $\tilde{z} \circ H$ does not already determine H. If M and M' are strictly pseudoconvex hypersurfaces in $\mathbb{C}^N$, then it follows from the work of Chern–Moser[16] that k_0 can be taken to be 2. Kruzhilin–Loboda[21] proved that for non-spherical strictly pseudoconvex hypersurfaces, the stability group can be linearized in Chern–Moser normal coordinates and, hence, one may take $k_0 = 1$. In both the spherical and non-spherical case, one can check directly that the k_0-jet of $\tilde{z} \circ H$ suffices to determine H.

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References

1 Baouendi M S, Ebenfelt P, Rothschild L P. Algebraicity of holomorphic mappings between real algebraic sets in $\mathbb{C}^n$. Acta Math, 1996, 177(2): 225–273
2 Baouendi M S, Ebenfelt P, Rothschild L P. Dynamics of the Segre varieties of a real submanifold in complex space. J Algebraic Geom, 2003, 12(1): 81–106
3 Webster S M. On the mapping problem for algebraic real hypersurfaces. Invent Math, 1977, 43(1): 53–68
4 Webster S M. On the reflection principle in several complex variables. Proc Amer Math Soc, 1978, 71(1): 26–28
5 Kohn J J. Boundary behavior of $\bar\partial$ on weakly pseudo-convex manifolds of dimension two. In: Collection of Articles Dedicated to Chern S S and Spencer D C on Their Sixtieth Birthdays. J Differential Geometry, 1972, 6: 523–542
6 Bloom T, Graham I. On "type" conditions for generic real submanifolds of $\mathbb{C}^n$. Invent Math, 1977, 40(3): 217–243
7 Ebenfelt P, Rothschild L P. Transversality of CR mappings. Amer J Math, in press
8 Zaitsev D. Germs of local automorphisms of real-analytic CR structures and analytic dependence on k-jets. Math Res Lett, 1997, 4(6): 823–842
9 Baouendi M S, Ebenfelt P, Rothschild L P. Convergence and finite determination of formal CR mappings. J Amer Math Soc, 2000, 13(4): 697–723
10 Baouendi M S, Mir N, Rothschild L P. Reflection ideals and mappings between generic submanifolds in complex space. J Geom Anal, 2002, 12(4): 543–580
11 Ebenfelt P, Lamel B, Zaitsev D. Finite jet determination of local analytic CR automorphisms and their parametrization by 2-jets in the finite type case. Geom Funct Anal, 2003, 13(3): 546–573
12 Kim S Y, Zaitsev D. Equivalence and embedding problems for CR-structures of any codimension. Topology, 2005, 44(3): 557–584
13 Lamel B, Mir N. Parametrization of local CR automorphisms by finite jets and applications. J Amer Math Soc, in press
14 Zaitsev D. Unique determination of local CR-maps by their jets: a survey. Harmonic analysis on complex homogeneous domains and Lie groups (Rome, 2001). Atti Accad. Naz. Lincei Cl Sci Fis Mat Natur Rend Lincei (9) Mat Appl, 2002, 13: 295–305
15 Baouendi M S. Strong unique continuation and finite jet determination for Cauchy-Riemann mappings. Progress in nonlinear differential equations and their applications. Birkhäuser: Verlag, in press
16 Chern S S, Moser J K. Real hypersurfaces in complex manifolds. Acta Math, 1974, 133: 219–271
17 Baouendi M S, Ebenfelt P, Rothschild L P. Real submanifolds in complex space and their mappings, volume 47 of Princeton Mathematical Series. Princeton: Princeton University Press, 1999
18 Baouendi M S, Ebenfelt P, Rothschild L P. Rational dependence of smooth and analytic CR mappings on their jets. Math Ann, 1999, 315(2): 205–249
19 D'Angelo J. Real hypersurfaces, orders of contact, and applications. Ann of Math, 1982, (3): 615–637
20 Lamel B, Mir N. Finite jet determination of CR mappings, preprint
21 Kruzhilin N G, Loboda A V. Linearization of local automorphisms of pseudoconvex surfaces. Dokl Akad Nauk SSSR, 271, English transl in Soviet Math, Dokl, 1983, 28: 70–72
22 Baouendi M S, Rothschild L P, Winkelmann J, et al. Lie group structures on groups of diffeomorphisms and applications to CR manifolds. Ann Inst Fourier (Grenoble), 2004, (5): 1279–1303
23 Baouendi M S, Ebenfelt P, Rothschild L P. Parametrization of local biholomorphisms of real analytic hypersurfaces. Asian J Math, 1997, 1(1): 1–16
24 Ebenfelt P, Lamel B. Finite jet determination of CR embeddings. J Geom Anal, 2004, 14(2): 241–265