Math. Ann. 315, 205–249 (1999)

Mathematische Annalen


# Rational dependence of smooth and analytic CR mappings on their jets

**M.S. Baouendi[1], P. Ebenfelt[2], Linda Preiss Rothschild[1]**

[1] Department of Mathematics, 0112, University of California at San Diego, La Jolla, CA 92093-0112, USA (e-mail: sbaouendi@ucsd.edu/lrothschild@ucsd.edu)
[2] Department of Mathematics, Royal Institute of Technology, S-100 44 Stockholm, Sweden (e-mail: ebenfelt@math.kth.se)

Received: 29 November 1998

*Mathematics Subject Classification (1991):* 32H02

## 0. Introduction

Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be two smooth ($C^\infty$) generic submanifolds with $p_0 \in M$ and $p_0' \in M'$. We shall consider holomorphic mappings $H : (\mathbb{C}^N, p_0) \to (\mathbb{C}^{N'}, p_0')$, defined in a neighborhood of $p_0 \in \mathbb{C}^N$, such that $H(M) \subset M'$ (and, more generally, smooth CR mappings $(M, p_0) \to (M', p_0')$; see below). We shall always work under the assumption that $M$ is of finite type at $p_0$ in the sense of Kohn and Bloom–Graham, and that $M'$ is finitely nondegenerate at $p_0'$ (see Sect. 1 for precise definitions). More precisely, we shall assume that $M'$ is $\ell_0$-nondegenerate at $p_0'$, for some integer $\ell_0 \geq 0$. (For a real hypersurface, 1-nondegeneracy at a point is equivalent to Levi nondegeneracy at that point.)

We denote by $J$ the complex structure map on $T\mathbb{C}^N$. Recall that for $p \in M$, $T_p^c M$ denotes the *complex tangent space* to $M$ at $p$, i.e. the largest $J$-invariant subspace of $T_p M$, the tangent space of $M$ at $p$. A smooth mapping $H : M \to M'$ is called *CR* if its tangent map $dH$ maps $T_p^c M$ into $T_{H(p)}^c M'$ for every $p \in M$. A CR mapping $H : M \to M'$ is called *CR submersive* at $p$ if $dH$ maps $T_p^c M$ onto $T_{H(p)}^c M'$. A holomorphic mapping $H$ sending $M$ into $M'$ is called CR submersive if its restriction to $M$ is. To a smooth CR

The first and the third authors are partially supported by National Science Foundation grant DMS 98-01258. The second author is partially supported by a grant from the Swedish Natural Science Research Council.

mapping $H : M \to M'$, and $p_0 \in M$, one may associate a unique formal (holomorphic) power series mapping

$$(0.1) \qquad \hat{H}(Z) \sim \sum_{\alpha} a_\alpha (Z - p_0)^\alpha, \quad a_\alpha \in \mathbb{C}^{N'},$$

which sends $(M, p_0)$ into $(M', p_0')$. (See Sect. 1 for details, definitions, and discussion.) If $H$ extends holomorphically to a neighborhood of $p_0$ in $\mathbb{C}^N$, then $\hat{H}(Z)$ is the Taylor series of $H$ at $p_0$. For $p \in \mathbb{C}^N$ and $p' \in \mathbb{C}^{N'}$, we shall denote by $J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(p,p')}$ the *jet space* of order $k$ of holomorphic mappings $(\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$. (See Sect. 2 for further details.)

Our main result gives rational dependence of the formal power series mapping associated to a CR submersive mapping, and more generally of a formal CR submersive power series mapping $(M, p_0) \to (M', p_0')$ (see Sect. 1 for precise definitions), on its jet of a predetermined order.

**Theorem 1.** *Let $M$ and $M'$ be smooth generic submanifolds through $p_0 \in \mathbb{C}^N$ and $p_0' \in \mathbb{C}^{N'}$, respectively, such that $M$ is of finite type at $p_0$ and $M'$ is $\ell_0$-nondegenerate at $p_0'$ for some integer $\ell_0 \geq 0$. Let $d$ be the codimension of $M$. Then there exist a finite number of formal power series mappings of the form*

$$(0.2) \qquad \Psi^k(Z, \Lambda) \sim \sum_{\alpha} \frac{Q_\alpha^k(\Lambda)}{P^k(\Lambda)^{l_\alpha^k}} (Z - p_0)^\alpha, \quad k = 1 \ldots, l,$$

*where $P^k$ and $Q_\alpha^k$ are ($\mathbb{C}$ and $\mathbb{C}^{N'}$ valued, respectively) polynomials on the jet space $J^{(d+1)\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(p_0,p_0')}$ and $l_\alpha^k$ are nonnegative integers, such that the following holds. For any smooth CR submersive mapping or, more generally, any formal (holomorphic) CR submersive power series mapping $\hat{H} : (M, p_0) \to (M', p_0')$, there exists $k \in \{1, \ldots, l\}$ with $P^k\left(j_{p_0}^{(d+1)\ell_0}(\hat{H})\right) \neq 0$ if $d+1$ is even and $P^k\left(\overline{j_{p_0}^{(d+1)\ell_0}(\hat{H})}\right) \neq 0$ if $d+1$ is odd, and for any such $k$,*

$$(0.3) \qquad \begin{aligned} \hat{H}(Z) &\sim \Psi^k\left(Z, j_{p_0}^{(d+1)\ell_0}(\hat{H})\right), \text{ if } d+1 \text{ is even}, \\ \hat{H}(Z) &\sim \Psi^k\left(Z, \overline{j_{p_0}^{(d+1)\ell_0}(\hat{H})}\right), \text{ if } d+1 \text{ is odd}. \end{aligned}$$

*If, in addition, $M$ and $M'$ are real-analytic, then the series* (0.2), *for $k \in \{1, \ldots, l\}$, converges in a neighborhood of $(p_0, \Lambda_0)$ in $\mathbb{C}^N \times J^{(d+1)\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(p_0,p_0')}$ for every $\Lambda_0$ satisfying $P^k(\Lambda_0) \neq 0$.*

Theorem 1, which will follow from the more general Theorem 2.1.5, has a number of applications. Our first application, which will be given in a more general form in Theorem 2.1.1, states that a holomorphic mapping sending $M$ into $M'$ which is CR submersive at $p_0$ is uniquely determined by finitely many derivatives at $p_0$.

**Theorem 2.** *Let $M$, $p_0$, $d$, $M'$, $p'_0$, and $\ell_0$ be as in Theorem 1. Then there exists an integer $k_0$, depending only on $M$, with $1 < k_0 \leq d+1$ such that the following holds. If $H^1, H^2 : (\mathbb{C}^N, p_0) \to (\mathbb{C}^{N'}, p'_0)$ are holomorphic mappings near $p_0$ such that $H^j(M) \subset M'$, $H^j$ is CR submersive at $p_0$, for $j = 1, 2$, and*

$$\frac{\partial^{|\alpha|} H^1}{\partial Z^\alpha}(p_0) = \frac{\partial^{|\alpha|} H^2}{\partial Z^\alpha}(p_0), \quad \forall \alpha : |\alpha| \leq k_0 \ell_0, \tag{0.4}$$

*then $H^1 \equiv H^2$.*

The conditions of finite type and finite nondegeneracy in Theorem 2 are also essentially necessary, in a certain sense, for the conclusion to hold. We refer the reader to the discussion in Sect. 2.2.

Our second application, Theorem 3 below (which is an easy consequence of Theorem 1), deals with real-analytic submanifolds. It gives sufficient conditions for all CR submersive formal mappings between real-analytic generic submanifolds to be convergent.

**Theorem 3.** *Let $M$ and $M'$ be real-analytic generic submanifolds through $p_0 \in \mathbb{C}^N$ and $p'_0 \in \mathbb{C}^{N'}$, respectively, such that $M$ is of finite type at $p_0$ and $M'$ is finitely nondegenerate at $p'_0$. Then, any formal (holomorphic) CR submersive mapping $\hat{H} : (M, p_0) \to (M, p'_0)$ is convergent, i.e. $\hat{H}$ is the Taylor series at $p_0$ of a holomorphic mapping $H : (\mathbb{C}^N, p_0) \to (\mathbb{C}^{N'}, p'_0)$ near $p_0$ with $H(M) \subset M'$. In particular, $(M, p_0)$ and $(M', p'_0)$ are formally equivalent if and only if they are biholomorphically equivalent.*

For our next application of Theorem 1, we shall denote the set of holomorphic mappings $H : (\mathbb{C}^N, p_0) \to (\mathbb{C}^{N'}, p'_0)$ which map $M$ into $M'$ and are CR submersive at $p_0$ by $\mathcal{F}(M, p_0; M', p'_0)$. This set has a natural inductive limit topology induced by uniform convergence on compact neighborhoods of $p_0$. We have the following result, which will be proved in Sect. 4.3.

**Theorem 4.** *Let $M$, $p_0$, $M'$, and $p'_0$ be as in Theorem 3. Let $d$ be the codimension of $M$, and $\ell_0$ a nonnegative integer such that $M'$ is $\ell_0$-nondegenerate at $p'_0$. Then there exist two real algebraic subvarieties*

$$A, B \subset J^{(d+1)\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(p_0, p'_0)} \tag{0.5}$$

*such that the mapping*

$$j_0^{(d+1)\ell_0} : \mathcal{F}(M, p_0; M', p'_0) \to J^{(d+1)\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(p_0, p'_0)} \tag{0.6}$$

*is a homeomorphism onto $A \setminus B$. In addition, the image $A \setminus B$ is totally real at each nonsingular point.*

The case where $(M,p_0) = (M',p_0')$ is of particular interest. In this case, the set of mappings $\mathcal{F}(M,p_0;M,p_0)$ consists of biholomorphisms of $(\mathbb{C}^N,p_0)$ (see Corollary 1.27) and, hence, forms a group under composition. This group is called the *stability group* of $M$ at $p_0$, and is denoted $\mathrm{Aut}(M,p_0)$. It follows from Theorem 4 that $\mathrm{Aut}(M,p_0)$, where $M$ is a real-analytic generic submanifold of codimension $d$ which is $\ell_0$-nondegenerate and of finite type at $p_0 \in M$, is a real Lie group which can be homeomorphically embedded as an algebraic, totally real subgroup of the jet group $G^{(d+1)\ell_0}(\mathbb{C}^N)_{p_0}$ (see also Theorem 2.1.14). Here $G^{(d+1)\ell_0}(\mathbb{C}^N)_{p_0}$ consists of those elements in $J^{(d+1)\ell_0}(\mathbb{C}^N,\mathbb{C}^N)_{(p_0,p_0)}$ which are invertible.

Recall that a real-analytic generic submanifold $M \subset \mathbb{C}^N$ is called *holomorphically nondegenerate* at $p \in M$ if there are no non-trivial holomorphic vector fields (i.e. holomorphic sections of the holomorphic tangent bundle $T'\mathbb{C}^N$) near $p$ which are tangent to $M$. If $M$ is connected, then it is either holomorphically nondegenerate at every point or at no point (see e.g. [BER4, Theorem 11.5.1]). We say that a connected real-analytic generic submanifold $M$ is holomorphically nondegenerate if it is so at some (and hence at every) point. The relation between holomorphic nondegeneracy and finite nondegeneracy is discussed in Sect. 2.2. The following result is then a corollary of Theorem 4 and the discussion in Sect. 2.2.

**Theorem 5.** *Let $M$ be a connected, real-analytic, holomorphically nondegenerate, generic submanifold of codimension $d$ in $\mathbb{C}^N$ which is of finite type at some point. Then there exists a proper real-analytic subvariety $V$ of $M$ such that the following holds for every $p \in M \setminus V$. The jet mapping*

$$j_p^{(d+1)(N-d)} : \mathrm{Aut}(M,p) \to G^{(d+1)(N-d)}(\mathbb{C}^N)_p$$

*is a continuous injective group homomorphism which is a homeomorphism onto a totally real algebraic Lie subgroup of $G^{(d+1)(N-d)}(\mathbb{C}^N)_p$.*

In Sect. 5, we consider smooth perturbations of generic submanifolds, satisfying the appropriate conditions, and study the behavior of the functions $\Psi^j$ in Theorem 1 under such perturbations (Theorem 5.1.1). As a consequence (Theorem 5.1.9) we obtain the result that if the stability group of a real-analytic generic submanifold $M$ is discrete, then it remains discrete under real-analytic (small) perturbations of $M$. One of the more important examples of a perturbation is allowing the base point $p_0 \in M$ to vary. As another application of Theorem 5.1.1, we show that the topological space $\cup_{p\in M,p'\in M'}\mathcal{F}(M,p;M',p')$ is homeomorphically embedded in the jet manifold $J^{(d+1)\ell_0}(\mathbb{C}^N,\mathbb{C}^{N'})$ (see Sect. 5.2) as a difference $A\setminus B$, where $A,B \subset J^{(d+1)\ell_0}(\mathbb{C}^N,\mathbb{C}^{N'})$ are real-analytic subvarieties whose fibers in $J^{(d+1)\ell_0}(\mathbb{C}^N,\mathbb{C}^{N'})_{(p,p')}$ are real-algebraic (see Theorem 5.2.9).

The paper concludes with an application, in Sect. 6, of our methods to the study of algebraicity of holomorphic mappings which map one real algebraic submanifold into another; the reader is referred to Theorem 6.1 for this result.

An important tool in the proofs of the results in this paper is the sequence of Segre mappings (see Sect. 3.1), which in the real-analytic case were introduced in [BER1] along with the Segre sets. The Segre variety of a real-analytic hypersurface $M$, which coincides with the first Segre set, was first introduced by Segre [Se]. Its use in the study of holomorphic mappings between real-analytic hypersurfaces was pioneered by Webster [W1], [W3]. Since then, its use has been crucial in the work of many mathematicians, including Diederich–Webster [DW], Diederich–Fornaess [DF], Forstneric [F], Huang [Hu], Diederich–Pinchuk [DP], and others. (See also the notes in [BER4, Chapters X–XII].) The Segre sets, introduced in [BER1] and playing a crucial role in the proofs in [BER1] and [BER2], have also been an important tool in the work of Zaitsev [Z2], [Z3]. We should also mention here the work of Christ, Nagel, Stein, and Wainger [CNSW] in a different context, in which they study the relation between certain curvature conditions on families of submanifolds in $\mathbb{R}^n$. Two of these conditions are strikingly similar to the two equivalent conditions in Theorem 3.1.9.

The study of automorphism groups of bounded domains in $\mathbb{C}^N$ goes back to H. Cartan [HC] (and was later continued by Kaneyuki [Ka] and, more recently, by Zaitsev [Z1]). The structure of the local transformation groups of Levi nondegenerate hypersurfaces in $\mathbb{C}^2$ was investigated by E. Cartan [EC1], [EC2] in connection with his work on the biholomorphic equivalence problem. His results were later extended to Levi nondegenerate hypersurfaces in higher dimensions by Tanaka [Ta] and Chern–Moser [CM]. In particular, the conclusions of Theorems 2,3, and 4 for real-analytic Levi nondegenerate hypersurfaces follow from their work. The convergence of the formal series (0.2) in Theorem 1 in the real-analytic case seems to be new even for Levi nondegenerate hypersurfaces. Further results on transformation groups of Levi nondegenerate hypersurfaces were obtained by a number of mathematicians, including Webster [W2], Burns-Shnider [BS], and the Moscow school (Beloshapka, Krushilin, Loboda, Vitushkin, etc.; see Krushilin [Kr], and Vitushkin [Vi]). Stanton [St1], [St2] considered infinitesimal CR automorphisms on general real-analytic hypersurfaces. (See also [BER2] for results on infinitesimal CR automorphism in higher codimensions.) The case of higher codimensions was considered by Tumanov–Henkin [TH], Tumanov [Tu] in the case of quadratic manifolds, and by Beloshapka [B] in the more general case where the Levi forms of the submanifolds are nondegenerate. For these classes of manifolds, the conclusion of Theorem 2 follows from their work.

We conclude the introduction by giving a brief history of results related to those of Theorems 1–4 above. Theorem 2, in the case $N = N'$ with $M$

and $M'$ real-analytic and of the same dimension, was obtained in [BER2]. In [BER3], Theorems 1, 3, and 4, with slightly weaker conclusions, were proved in the case $N = N'$ with $M$ and $M'$ real-analytic hypersurfaces. Zaitsev [Z2] proved a weaker version of Theorem 1 for real-analytic CR submersive mappings (in the real-analytic case), namely one in which the jet space of order $(d+1)\ell_0$ is replaced by that of order $2(d+1)\ell_0$ and where the dependence on the jets is only local analytic instead of rational. In particular, his result shows that the stability group is a Lie group with the natural topology. However, for the application given by Theorem 3 it is important to prove Theorem 1 for formal CR submersive mappings.

For most of the proofs of the results mentioned above, it is convenient to work with formal mappings between formal generic submanifolds. Hence, most results presented here will be reformulated, and proved, in this more general context. The following section presents the necessary preliminaries and definitions. In what follows, the distinguished points $p_0$ and $p_0'$ on $M$ and $M'$, respectively, will, for convenience and without loss of generality, be assumed to be 0.

## 1. Preliminaries on formal submanifolds and mappings

Let $\mathbb{C}[[x]] = \mathbb{C}[[x_1, \dots, x_k]]$ be the ring of formal power series in $x = (x_1, \dots, x_k)$ with complex coefficients. Suppose that $\rho = (\rho_1, \dots, \rho_d) \in \mathbb{C}[[Z, \zeta]]^d$, where $Z = (Z_1, \dots, Z_N)$ and $\zeta = (\zeta_1, \dots, \zeta_N)$, satisfies the reality condition

$$(1.1) \qquad \rho(Z, \zeta) \sim \bar{\rho}(\zeta, Z),$$

where $\bar{\rho}$ is the formal series obtained from $\rho$ by replacing each coefficient in the series by its complex conjugate; we use the symbol $\sim$ to denote equality of formal power series. If, in addition, the series $\rho$ satisfies the condition $\rho(0) = 0$, and

$$(1.2) \qquad \partial\rho_1(0) \wedge \ldots \wedge \partial\rho_d(0) \neq 0,$$

then we say that $\rho$ defines a *formal real submanifold* $M$ of $\mathbb{C}^N$ through 0 of codimension $d$ (and dimension $2N - d$). If $M'$ is another such formal real submanifold defined by $\rho' = (\rho_1', \dots, \rho_d')$, then we shall say that $M = M'$ if there exists a $d \times d$ matrix of formal power series $a(Z, \zeta)$ (necessarily invertible at 0) such that

$$(1.3) \qquad \rho(Z, \zeta) \sim a(Z, \zeta)\rho'(Z, \zeta).$$

These definitions are motivated by the fact that if in addition the components of $\rho$ are convergent power series, then the equations $\rho(Z, \bar{Z}) = 0$

define a real-analytic submanifold $M$ of $\mathbb{C}^N$ through 0. Moreover, if $M'$ is another such defined by a convergent power series $\rho'$, then (1.3) holds if and only if $M$ and $M'$ are the same. Also, if $M$ is a smooth real submanifold in $\mathbb{C}^N$ through 0, then the Taylor series at 0 of a smooth defining function $\rho(Z,\bar{Z})$ of $M$ near 0, with $\bar{Z}$ formally replaced by $\zeta$, defines a formal real submanifold through 0. These observations will be used to deduce the results given in the introduction from the corresponding results for formal real submanifolds.

If the formal series $\rho$ defining $M$ satisfies the stronger condition

$$\partial_Z\rho_1(0)\wedge\ldots\wedge\partial_Z\rho_d(0)\neq 0, \tag{1.4}$$

(which in particular implies (1.2)) then we say that the formal real submanifold $M$ is *generic*. We say that a formal vector field

$$X=\sum_{j=1}^{N}\left(a_j(Z,\zeta)\frac{\partial}{\partial Z_j}+b_j(Z,\zeta)\frac{\partial}{\partial \zeta_j}\right), \tag{1.5}$$

with $a_j, b_j\in\mathbb{C}[[Z,\zeta]]$, is tangent to the formal real submanifold $M$ if

$$X\rho(Z,\zeta)\sim a(Z,\zeta)\rho(Z,\zeta), \tag{1.6}$$

for some $d\times d$ matrix of formal power series $a(Z,\zeta)$.

We say that the formal vector field $X$ in (1.6) is of type $(0,1)$ if $a_j\sim 0$, $j=1,\ldots,N$, and similarly of type $(1,0)$ if the $b_j\sim 0$. Let $\mathcal{D}_M$ denote the $\mathbb{C}[[Z,\zeta]]$-module generated by all formal $(0,1)$ and $(1,0)$ vector fields tangent to $M$, and $\mathfrak{g}_M$ the Lie algebra generated by $\mathcal{D}_M$. We denote by $\mathfrak{g}_M(0)$ the complex vector space obtained by evaluating the coefficients of the formal vector fields in $\mathfrak{g}_M$ at 0. Similarly, we use the notation $\mathcal{D}_M(0)$ for the complex vector space obtained by evaluating the coefficients of the formal vector fields in $\mathcal{D}_M$ at 0. (The reader should observe the analogy, for a smooth real submanifold $M$ through 0, between the complexified complex tangent space $\mathbb{C}T_0^cM$ to $M$ at 0 and $\mathcal{D}_M(0)$ for the corresponding formal submanifold.) Thus, we have $\mathcal{D}_M(0)\subset\mathfrak{g}_M(0)\subset T_0'\mathbb{C}^{2N}$, where $T_0'\mathbb{C}^{2N}$ denotes the holomorphic tangent space of $\mathbb{C}^{2N}$ at 0.

We say that $M$ is of *finite type* at 0 if $\dim_{\mathbb{C}}\mathfrak{g}_M(0)=\dim M=2N-d$. (Note that the vector space $\mathcal{D}_M(0)$ has dimension $2N-2d$; this follows easily from the fact that $M$ is generic and of codimension $d$.)

We shall also need the notion of finite nondegeneracy of a formal generic submanifold. We say that the formal generic submanifold $M$ is *finitely nondegenerate* at 0 if there exists an integer $\ell\geq 0$ such that

$$\operatorname{span}\left\{L^\alpha\left(\frac{\partial\rho_j}{\partial Z}\right)(0):1\leq j\leq d,\ |\alpha|\leq\ell\right\}=\mathbb{C}^N. \tag{1.7}$$

Here, $L_1, \ldots, L_n$ is a basis for the $\mathbb{C}[[Z,\zeta]]$-module of all formal $(0,1)$ vector fields tangent to $M$ (so $n = N - d$) modulo those whose coefficients are in the ideal generated by $\rho_1, \ldots, \rho_d$. We also use multi-index notation, i.e. we introduce the vector $L = (L_1, \ldots, L_n)$ and, for any $\alpha \in \mathbb{Z}_+^n$, we write

$$L^\alpha = L_1^{\alpha_1} \ldots L_n^{\alpha_n}, \quad |\alpha| = \sum_{j=1}^{n} \alpha_j. \tag{1.8}$$

More precisely, we say that $M$ is *$\ell_0$-nondegenerate* at 0 if $\ell_0$ is the smallest integer for which (1.7) holds. It is an easy exercise to show that the definition of $\ell_0$-nondegeneracy (and hence that of finite nondegeneracy) does not depend on the choice of basis $L = (L_1, \ldots, L_n)$, defining series $\rho = (\rho_1, \ldots, \rho_d)$, or the choice of coordinates $Z$. Hence, $\ell_0$-nondegeneracy is a property of the formal generic submanifold $M$. The reader is also referred to [BER4] for further discussion of these notions, as well as that of finite type, for smooth and real-analytic generic submanifolds.

Let $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ be a formal mapping, i.e. $H \in \mathbb{C}[[Z_1, \ldots, Z_N]]^{N'}$ such that each component of $H(Z) = (H_1(Z), \ldots, H_{N'}(Z))$ has no constant term. To such a formal mapping $H$ we associate a formal mapping $\mathcal{H} : (\mathbb{C}^N \times \mathbb{C}^N, 0) \to (\mathbb{C}^{N'} \times \mathbb{C}^{N'}, 0)$ defined by

$$\mathcal{H}(Z,\zeta) \sim (H(Z), \bar{H}(\zeta)). \tag{1.9}$$

If $M$ and $M'$ are formal real submanifolds of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ defined by formal series $\rho(Z,\zeta) = (\rho_1(Z,\zeta), \ldots, \rho_d(Z,\zeta))$ and $\rho'(Z',\zeta') = (\rho_1'(Z',\zeta'), \ldots, \rho_{d'}'(Z',\zeta'))$, respectively, then we say that the formal mapping $H$, as above, maps $M$ into $M'$, denoted $H : (M,0) \to (M',0)$, if

$$\rho'(H(Z), \bar{H}(\zeta)) \sim c(Z,\zeta)\rho(Z,\zeta), \tag{1.10}$$

for some $d' \times d$ matrix $c(Z,\zeta)$ of formal power series. It will be convenient to choose *normal coordinates*, $Z = (z,w)$ and $\zeta = (\chi,\tau)$ with $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_d)$ (so $n + d = N$), $\chi = (\chi_1, \ldots, \chi_n)$, and $\tau = (\tau_1, \ldots, \tau_d)$, in $\mathbb{C}^N \times \mathbb{C}^N$ for $M$ at 0. By this we mean a formal change of coordinates $Z = Z(z,w)$ with $Z(z,w)$ a formal invertible mapping $(\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$, and $\zeta = \bar{Z}(\chi,\tau)$ the corresponding change, such that

$$\rho(Z(z,w), \bar{Z}(\chi,\tau)) \sim a(z,w,\chi,\tau)(w - Q(z,\chi,\tau)), \tag{1.11}$$

where $a(z,w,\chi,\tau)$ is an invertible $d \times d$ matrix of formal power series, and the vector valued $Q \in \mathbb{C}[[z,\chi,\tau]]^d$ satisfies

$$Q_j(0,\chi,\tau) \sim Q_j(z,0,\tau) \sim \tau_j, \quad j = 1, \ldots, d. \tag{1.12}$$

(See [BER4, Chapter IV.2].) It follows from the reality of $\rho$ that

$$\rho(Z(z,w),\bar{Z}(\chi,\tau)) \sim b(z,w,\chi,\tau)(\tau-\bar{Q}(\chi,z,w)), \tag{1.13}$$

where $b(z,w,\chi,\tau)$ is an invertible $d\times d$ matrix of formal power series. Similarly, we choose normal coordinates $Z'=(z',w')$ and $\zeta'=(\chi',\tau')$ in $\mathbb{C}^{N'}\times\mathbb{C}^{N'}$, with $z'=(z'_1,\dots,z'_{n'})$ and $w'=(w'_1,\dots,w'_{d'})$ (so $n'+d'=N'$), such that $M'$ is defined by $w'-Q'(z',\chi',\tau')$ (or, more precisely, by $a'(z,w,\chi,\tau)(w'-Q'(z',\chi',\tau'))$, for some matrix $a'$ making the expression real). Then we may write the formal mapping $H=(F,G)$, with $F=(F_1,\dots,F_{n'})$ and $G=(G_1,\dots,G_{d'})$, and the condition $H:(M,0)\to(M',0)$ can be expressed by either of the equations

$$\begin{gathered} G(z,w)\sim Q'(F(z,w),\bar{F}(\chi,\tau),\bar{G}(\chi,\tau)) \\ \text{or} \\ \bar{G}(\chi,\tau)\sim\bar{Q}'(\bar{F}(\chi,\tau),F(z,w),G(z,w)), \end{gathered} \tag{1.14}$$

for $\tau=\bar{Q}(\chi,z,w)$ or $w=Q(z,\chi,\tau)$. An observation that will be useful is that $G(z,0)\sim 0$, as is easily verified by taking $w=\tau=0$ and $\chi=0$ in (1.14).

Note that if $M$ and $M'$ correspond to real-analytic submanifolds and the formal mapping $H$ defines a holomorphic mapping $(\mathbb{C}^N,0)\to(\mathbb{C}^{N'},0)$ in some neighborhood of the origin, then $H:(M,0)\to(M',0)$ if and only if $H(M)\subset M'$. Moreover, if $M\subset\mathbb{C}^N$ and $M'\subset\mathbb{C}^{N'}$ are smooth CR submanifolds through the origin and $h:M\to M'$ is a smooth CR mapping, with $h(0)=0$, then there exists a unique formal mapping $H:(M,0)\to(M',0)$ such that, for any local parametrization

$$\mathbb{R}^{2N-d}\supset U\ni x\mapsto Z(x)\in M, \tag{1.15}$$

with $Z(0)=0$, we have

$$h(Z(x))\sim H(Z(x)), \tag{1.16}$$

where the left hand side of (1.16) refers to the Taylor expansion at 0 of the smooth mapping $x\mapsto h(Z(x))$, and the right hand side is taken in the sense of composition of $H$ and the Taylor series at 0 of $Z(x)$. (See e.g. [BER4, Sect. 1.7]). Observe that if $h$ is the restriction to $M$ of a holomorphic mapping $(\mathbb{C}^N,0)\to(\mathbb{C}^{N'},0)$, then $H(Z)$ is the Taylor series of this holomorphic mapping.

Given a formal mapping $H:(\mathbb{C}^N,0)\to(\mathbb{C}^{N'},0)$, we denote the tangent mapping of the associated formal mapping $\mathcal{H}:(\mathbb{C}^N\times\mathbb{C}^N,0)\to(\mathbb{C}^{N'}\times\mathbb{C}^{N'},0)$ by $d\mathcal{H}:T'_0\mathbb{C}^{2N}\to T'_0\mathbb{C}^{2N'}$; thus, $d\mathcal{H}$ is the mapping taking a vector

$$X=\sum_{j=1}^{N}\left(a_j\frac{\partial}{\partial Z_j}+b_j\frac{\partial}{\partial\zeta_j}\right),\quad a_j,b_j\in\mathbb{C} \tag{1.17}$$

in $T_0'\mathbb{C}^N$ to the vector

$$(1.18)\qquad d\mathcal{H}(X)=\sum_{j=1}^{N'}\left((XH_j(Z))(0)\frac{\partial}{\partial Z_j'}+(X\bar{H}_j(\zeta))(0)\frac{\partial}{\partial \zeta_j'}\right)$$

in $T_0'\mathbb{C}^{N'}$. Recall that we say that the vector $X$ in (1.17) is a $(1,0)$ vector if $b_j=0$, $j=1,\dots,N$, and $(0,1)$ vector if $a_j=0$, $j=1,\dots,N$. It is clear that $d\mathcal{H}$ maps $(0,1)$ vectors to $(0,1)$ vectors and $(1,0)$ vectors to $(1,0)$ vectors. If $H:(M,0)\to(M',0)$, then it follows that $d\mathcal{H}$ maps $\mathcal{D}_M(0)$ into $\mathcal{D}_{M'}(0)$. We shall say that a formal mapping $H:(M,0)\to(M',0)$ is *CR submersive* at 0 if

$$(1.19)\qquad d\mathcal{H}(\mathcal{D}_M(0))=\mathcal{D}_{M'}(0).$$

**Proposition 1.20.** *Let $M$ and $M'$ be formal generic submanifolds through the origin in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively. If $H:(M,0)\to(M',0)$ is a CR submersive formal mapping, then $d\mathcal{H}(\mathfrak{g}_M(0))=\mathfrak{g}_{M'}(0)$.*

*Proof.* We denote by $(\rho)$ the ideal in $\mathbb{C}[[Z,\zeta]]$ generated by $\rho_1,\dots,\rho_d$, a set of defining formal series for $M$, and define similarly the ideal $(\rho')$ in $\mathbb{C}[[Z',\zeta']]$ corresponding to $M'$. Let $L_1',\dots,L_{n'}'$ be a basis for the formal $(1,0)$ vector fields tangent to $M'$ modulo those whose coefficients are in the ideal $(\rho')$ and $\bar{L}_1',\dots,\bar{L}_{n'}'$ a basis, modulo $(\rho')$, for the formal $(0,1)$ vector fields tangent to $M'$. We claim that for any formal $(1,0)$ vector field $L$ tangent to $M$ there exist formal power series $a_j=a_j(Z,\zeta)$, $j=1,\dots n'$, such that for any $f\in\mathbb{C}[[Z',\zeta']]$ we have

$$(1.21)\qquad L(f\circ\mathcal{H})\sim\sum_{j=1}^{n'}a_j((L_j'f)\circ\mathcal{H})\mod(\rho)$$

as power series in $(Z,\zeta)$. To prove the claim, it suffices to find $a_j(Z,\zeta)$ satisfying (1.21) with $f(Z',\zeta')=Z_k'$, $k=1,\dots N'$. This is done by using the chain rule and elementary linear algebra. The details are left to the reader. Similarly, for any $(0,1)$ vector field $\bar{L}$ tangent to $M$, we can find $\tilde{a}_j(Z,\zeta)$ such that

$$(1.22)\qquad \bar{L}(f\circ\mathcal{H})\sim\sum_{j=1}^{n'}\tilde{a}_j((\bar{L}_j'f)\circ\mathcal{H})\mod(\rho)$$

as power series in $(Z,\zeta)$. Thus, for bases, modulo $(\rho)$, $L_1,\dots,L_n$ of the $(1,0)$ vector fields tangent to $M$ and $\bar{L}_1,\dots,\bar{L}_n$ of the $(0,1)$ vector fields tangent to $M$, we obtain two $n \times n'$ matrices $(a_{jk}(Z,\zeta))$ and $(\tilde{a}_{jk}(Z,\zeta))$ of formal power series such that

$$(1.23)\qquad \begin{aligned} L_j(f\circ\mathcal{H}) &\sim \sum_{k=1}^{n'} a_{jk}((L'_k f)\circ\mathcal{H}) \mod (\rho), \\ \bar{L}_j(f\circ\mathcal{H}) &\sim \sum_{k=1}^{n'} \tilde{a}_{jk}((\bar{L}'_k f)\circ\mathcal{H}) \mod (\rho), \end{aligned}$$

for all $f \in \mathbb{C}[[Z',\zeta']]$. It is easy to verify that (1.19) implies that the rank of each of these matrices at 0 equals $n'$. Hence, we may assume, after a linear transformation of the $L_j$'s and $\bar{L}_j$'s (over $\mathbb{C}[[Z,\zeta]]$) if necessary, that for $j = 1,\ldots,n'$

$$(1.24)\qquad \begin{aligned} L_j(f\circ\mathcal{H}) &\sim (L'_j f)\circ\mathcal{H} \mod (\rho), \\ \bar{L}_j(f\circ\mathcal{H}) &\sim (\bar{L}'_j f)\circ\mathcal{H} \mod (\rho), \end{aligned}$$

for all $f \in \mathbb{C}[[Z',\zeta']]$. It follows immediately from (1.24) that we also have

$$(1.25)\qquad [X,Y](f\circ\mathcal{H}) \sim ([X',Y']f)\circ\mathcal{H} \mod (\rho),$$

for any $X, Y \in \{L_1,\ldots,L_{n'},\bar{L}_1,\ldots,\bar{L}_{n'}\}$ and corresponding $X', Y' \in \{L'_1,\ldots,L'_{n'},\bar{L}'_1,\ldots,\bar{L}'_{n'}\}$ (i.e. such that $X, X'$ and $Y, Y'$ satisfy (1.24)). In particular, we have

$$(1.26)\qquad d\mathcal{H}([X,Y]_0) = [X',Y']_0.$$

Repeating this argument for commutators of any length, we can conclude that $d\mathcal{H}(\mathfrak{g}_M(0)) = \mathfrak{g}_{M'}(0)$. This completes the proof of Proposition 1.20. $\square$

**Corollary 1.27.** *Let $M$ and $M'$ be formal generic submanifolds of codimension $d$ and $d'$ through the origin in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively. Assume that $M'$ is of finite type at 0, and let $H : (M,0) \to (M',0)$ be a CR submersive formal mapping. Then $dH(T'_0\mathbb{C}^N) = T'_0\mathbb{C}^{N'}$ and $d \geq d'$. If in addition $N = N'$, then $dH$ is an isomorphism of $T'_0\mathbb{C}^N$ into itself, i.e. the formal mapping $H$ is invertible.*

*Proof.* Since $d\mathcal{H}$ maps $\mathcal{D}_M(0)$ onto $\mathcal{D}_{M'}(0)$ by assumption, and hence $\mathfrak{g}_M(0)$ onto $\mathfrak{g}_{M'}(0)$ by Proposition 1.20, it follows that the induced mapping from the quotient space $\mathfrak{g}_M(0)/\mathcal{D}_M(0)$ is onto $\mathfrak{g}_{M'}(0)/\mathcal{D}_{M'}(0)$. Since $M'$ is of finite type at 0, it follows that $\dim \mathfrak{g}_{M'}(0)/\mathcal{D}_{M'}(0) = d'$. Since $d \geq \dim \mathfrak{g}_M(0)/\mathcal{D}_M(0)$, it follows that $d \geq d'$. (Hence, we also have $N \geq N'$.)

To prove that $dH(T_0'\mathbb{C}^N) = T_0'\mathbb{C}^{N'}$, we take normal coordinates $Z = (z, w)$, $\zeta = (\chi, \tau)$ in $\mathbb{C}^{2N}$ for $M$ and $Z' = (z', w')$, $\zeta' = (\chi', \tau')$ in $\mathbb{C}^{2N'}$ for $M'$. For $H = (F, G)$, the fact that $H : (M, 0) \to (M', 0)$ is expressed by (1.14). Also, $d\mathcal{H}(\mathcal{D}_M(0)) = \mathcal{D}_{M'}(0)$ is equivalent to $\partial F/\partial z(0)$ having rank $n'$. Using the facts that $G(z, 0) \sim 0$ and $M'$ is of finite type at 0, and applying Proposition 1.20, we conclude that the rank of $\partial G/\partial w(0)$ is $d'$. This completes the proof of $dH(T_0'\mathbb{C}^N) = T_0'\mathbb{C}^{N'}$. The second statement of Corollary 1.27 is an immediate consequence of the first. □

## 2. Uniqueness and parametrization of formal mappings

### *2.1. Main results*

In this section, we shall give results on uniqueness and parametrization of formal mappings between formal real submanifolds from which Theorems 1–3 presented in the introduction will follow. We first give sufficient conditions so that a mapping sending $M$ into $M'$ is determined by a finite number of derivatives of the mapping at 0. The necessity of these conditions will be discussed in Sect. 2.2.

**Theorem 2.1.1.** *Let $M$ and $M'$ be formal generic submanifolds through $0 \in \mathbb{C}^N$ and $0 \in \mathbb{C}^{N'}$, respectively, such that $M$ is of finite type at 0 and $M'$ is $\ell_0$-nondegenerate at 0 for some integer $\ell_0$. Let $d$ denote the codimension of $M$. Then there exists an integer $k_0$, depending only on $M$, with $1 < k_0 \leq d + 1$, such that the following holds. If $H^1, H^2 : (M, 0) \to (M', 0)$ are CR submersive formal mappings such that*

$$\text{(2.1.2)} \qquad \frac{\partial^{|\alpha|} H^1}{\partial Z^\alpha}(0) = \frac{\partial^{|\alpha|} H^2}{\partial Z^\alpha}(0), \quad \forall \alpha : |\alpha| \leq k_0 \ell_0,$$

*then $H^1 \sim H^2$.*

*Remark* 2.1.3. If $N = N'$, $\dim M = \dim M'$, and $H^j$, $j = 1, 2$, are invertible formal mappings, then $d\mathcal{H}^j : \mathcal{D}_M(0) \to \mathcal{D}_{M'}(0)$, $j = 1, 2$, are necessarily isomorphisms, and hence surjective. More generally, if $n = n'$ (recall that $n$ denotes $N - d$, where $d$ denotes the codimension of $M$, and similarly for $n'$ and $M'$) and the mappings $d\mathcal{H}^j : \mathcal{D}_M(0) \to \mathcal{D}_{M'}(0)$, $j = 1, 2$, are injective, then they are also necessarily surjective. (Indeed, $\dim \mathcal{D}_M(0) = 2n$.)

It is clear from the remarks in Sect. 1 that Theorem 2.1.1 is a more general version of Theorem 2 in the introduction. The proof of Theorem 2.1.1 will be given in Sect. 3.4.

Let $E(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$ denote the set of germs of holomorphic mappings $(\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ and $\hat{E}(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$ the set of formal mappings $(\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$. For each positive integer $k$, we denote by $J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$ the jet space of order $k$ of holomorphic mappings $(\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$, and by $j_0^k : \hat{E}(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)} \to J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$ the jet mapping taking a formal mapping $H$ to its $k$th jet at 0, $j_0^k(H)$. In particular, $J^1(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$ can be viewed as the space of linear mappings from $\mathbb{C}^N$ to $\mathbb{C}^{N'}$. For $k \geq l \geq 1$, we denote by

$$j_0^{k,l} : J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)} \to J^l(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)} \tag{2.1.4}$$

the canonical mapping induced by $j_0^l : \hat{E}(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)} \to J^l(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$.

Given coordinates $Z$ and $Z'$ on $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, the jet space $J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$ can be identified with the set of polynomial mappings $(\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ of degree $k$. The coordinates on $J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$, which we will denote by $\Lambda$, can then be taken to be the coefficients of these polynomials. Observe that formal changes of coordinates in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ give a polynomial change of coordinates in $J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$.

The reader is referred e.g. to [GG] or [BER4, Chapter XII] for further discussion of these notions. Our main result is the following, which in particular implies Theorem 2.1.1 above and Theorem 1 in the introduction (we leave the details of the proofs of these implications to the reader), and from which several other theorems will be deduced below.

**Theorem 2.1.5.** *Let $M$ and $M'$ be formal generic submanifolds through $0 \in \mathbb{C}^N$ and $0 \in \mathbb{C}^{N'}$ of codimension $d$ and $d'$, respectively, such that $M$ is of finite type at 0 and $M'$ is $\ell_0$-nondegenerate at 0 for some integer $\ell_0$. Assume that $n \geq n'$, where $n = N - d$ and $n' = N' - d'$. Then there exists an integer $k_1$ with $1 < k_1 \leq d + 1$, such that for each $\tilde{\jmath} = (j_1, \ldots, j_{n'})$, with $1 \leq j_1 < \ldots < j_{n'} \leq n$, there exists a polynomial $P^{\tilde{\jmath}}$ on $J^{k_1\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$ and a formal power series in $Z = (Z_1, \ldots, Z_N)$*

$$\Phi^{\tilde{\jmath}}(Z, \Lambda) \sim \sum_{|\alpha|>0} \frac{c_\alpha^{\tilde{\jmath}}(\Lambda)}{P^{\tilde{\jmath}}(\Lambda)^{l_\alpha^{\tilde{\jmath}}}} Z^\alpha, \tag{2.1.6}$$

*where $c_\alpha^{\tilde{\jmath}}(\Lambda)$ are $\mathbb{C}^{N'}$ valued polynomials on $J^{k_1\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$ and $l_\alpha^{\tilde{\jmath}}$ nonnegative integers, satisfying the following. For every formal mapping $H\colon (M, 0) \to (M', 0)$, which is CR submersive i.e.*

$$d\mathcal{H}(\mathcal{D}_M(0)) = \mathcal{D}_{M'}(0), \tag{2.1.7}$$

*there exists $\tilde{\jmath}$ as above such that $P^{\tilde{\jmath}}(j_0^{k_1\ell_0}(H)) \neq 0$ when $k_1$ is even, $P^{\tilde{\jmath}}(\overline{j_0^{k_1\ell_0}(H)}) \neq 0$ when $k_1$ is odd, and*

$$(2.1.8) \qquad \begin{aligned} H(Z) &\sim \Phi^{\tilde{\jmath}}(Z, j_0^{k_1\ell_0}(H)), \text{ if } k_1 \text{ is even,} \\ H(Z) &\sim \Phi^{\tilde{\jmath}}\left(Z, \overline{j_0^{k_1\ell_0}(H)}\right), \text{ if } k_1 \text{ is odd.} \end{aligned}$$

*In addition, if $M$ and $M'$ are real-analytic, then for every $\tilde{\jmath}$ as above and $\Lambda_0$ with $P^{\tilde{\jmath}}(\Lambda_0) \neq 0$ the series* (2.1.6) *converges uniformly for $(Z, \Lambda)$ near $(0, \Lambda_0)$ in $\mathbb{C}^N \times J^{k_1\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$.*

In what follows, we shall denote by $\hat{\mathcal{F}}(M, M') = \hat{\mathcal{F}}(M, 0; M', 0)$ the set of formal mappings $(M, 0) \to (M', 0)$ which are CR submersive. (For brevity, we here suppress the dependence on the base points of $M$ and $M'$, which in this section are assumed to be the origin in the respective spaces.) When $M$ and $M'$ are real-analytic, then we also denote by $\mathcal{F}(M, M') := \mathcal{F}(M, 0; M', 0)$ those formal mappings in $\hat{\mathcal{F}}(M, M')$ that are convergent, and hence define holomorphic mappings which map a neighborhood of 0 in $M$ into a neighborhood of 0 in $M'$. Thus, in this notation, Theorem 3 in the introduction gives sufficient conditions on $M$ and $M'$ so that $\mathcal{F}(M, M') = \hat{\mathcal{F}}(M, M')$.

The following result, which will be proved in Sect. 4.3, is based on Theorem 2.1.5.

**Theorem 2.1.9.** *Let $M$ and $M'$ be formal generic submanifolds through $0 \in \mathbb{C}^N$ and $0 \in \mathbb{C}^{N'}$, respectively, such that $M$ is of finite type at 0 and $M'$ is $\ell_0$-nondegenerate at 0 for some integer $\ell_0$. Then there exist an integer $k_1$, depending only on $M$, with $1 < k_1 \leq d+1$ where $d$ denotes the codimension of $M$, and two real algebraic subvarieties $A$, $B \subset J^{k_1\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$ such that the image of the mapping*

$$(2.1.10) \qquad j_0^{k_1\ell_0} : \hat{\mathcal{F}}(M, M') \to J^{k_1\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$$

*coincides with $A \setminus B$. In addition, the image $A \setminus B$ is totally real at each nonsingular point.*

*Remark* 2.1.11. If $M$ and $M'$ are real-analytic, then, in view of Theorem 3, the conclusion of Theorem 2.1.9 also holds for convergent maps, i.e. with $\hat{\mathcal{F}}(M, M')$ replaced by $\mathcal{F}(M, M')$ in (2.1.10). In this case, (2.1.10) is a homeomorphism onto its image. This is the content of Theorem 4, which will be proved in Sect. 4.3.

Let us consider the case $N = N'$ and $M = M'$, where $M$ is of finite type at 0. By Corollary 1.27, the set $\hat{\mathcal{F}}(M, M)$ consists of formal mappings

$H : (M,0) \to (M,0)$ which are invertible. Thus, $\hat{\mathcal{F}}(M,M)$ is a group under composition.

For $k \geq 1$, we denote by $G^k(\mathbb{C}^N)_0$ the group (under composition) of invertible jets in $J^k(\mathbb{C}^N,\mathbb{C}^N)_{(0,0)}$, which is a complex Lie group. It follows from the above that, for any $k \geq 1$, the image of $\hat{\mathcal{F}}(M,M)$ under $j_0^k$ is contained in $G^k(\mathbb{C}^N)_0 \subset J^k(\mathbb{C}^N,\mathbb{C}^N)_{(0,0)}$.

**Theorem 2.1.12.** *Let $M$ be a formal generic submanifold through 0 in $\mathbb{C}^N$ which is $\ell_0$-nondegenerate and of finite type at 0. Then there exists an integer $k_1$ with $1 < k_1 \leq d+1$, where $d$ denotes the codimension of $M$, such that*

$$j_0^{k_1\ell_0} : \hat{\mathcal{F}}(M,M) \to G^{k_1\ell_0}(\mathbb{C}^N)_0 \tag{2.1.13}$$

*is an injective group homomorphism and its image is a totally real algebraic Lie subgroup of $G^{k_1\ell_0}(\mathbb{C}^N)_0$.*

If $M$ is a real-analytic generic submanifold through 0 in $\mathbb{C}^N$, then, as mentioned in the introduction, $\mathcal{F}(M,M)$ (which in view of Theorem 3 coincides with $\hat{\mathcal{F}}(M,M)$) is usually called the stability group of $M$ at 0, and is denoted by $\mathrm{Aut}(M,0)$. The group $\mathrm{Aut}(M,0)$ has a natural (inductive limit) topology corresponding to uniform convergence on compact neighborhoods of 0. That is, a sequence $\{H_j\} \subset \mathrm{Aut}(M,0)$ converges to $H \in \mathrm{Aut}(M,0)$ if there is a compact neighborhood of 0 to which all the $H_j$ extend and on which the $H_j$ converge uniformly to $H$.

**Theorem 2.1.14.** *Let $M$ be a real-analytic generic submanifold through 0 in $\mathbb{C}^N$ which is $\ell_0$-nondegenerate and of finite type at 0. Then there exists an integer $k_1$ with $1 < k_1 \leq d+1$, where $d$ is the codimension of $M$, such that*

$$j_0^{k_1\ell_0} : \mathrm{Aut}(M,0) \to G^{k_1\ell_0}(\mathbb{C}^N)_0 \tag{2.1.15}$$

*is a continuous injective group homomorphism and its image is a totally real algebraic Lie subgroup of $G^{k_1\ell_0}(\mathbb{C}^N)_0$. Moreover,* (2.1.15) *is a homeomorphism onto the image $j_0^{k_1\ell_0}(\mathrm{Aut}(M,0))$.*

For the proofs of the results above, we shall need several tools which will be presented in Sect. 3 below. However, we first discuss briefly the necessity of the conditions imposed on $M$ and $M'$ in the results above.

### *2.2. Generic necessity of finite type and finite nondegeneracy in the real-analytic case*

In the theorems given in Sect. 2.1, a standing assumption is that $M$ is of finite type at 0 and that $M'$ is finitely nondegenerate at 0. In this section, we shall discuss to what extent these conditions are necessary for the results. More precisely, we shall discuss failure of the conclusion in Theorem 2.1.1 (which is a consequence of the main result, Theorem 2.1.5).

The notion of finite nondegeneracy at a point $p$ in a real-analytic, generic submanifold $M$ is intimately related to that of holomorphic nondegeneracy as defined in the introduction. A connected, real-analytic, generic submanifold $M \subset \mathbb{C}^N$ of codimension $d$ is holomorphically nondegenerate (at some point or, equivalently, at all points) if and only there exists $\ell(M)$, $0 \leq \ell(M) \leq N-d$, such that $M$ is $\ell(M)$-nondegenerate outside a proper real-analytic subvariety of $M$ (see e.g. [BER1] or [BER4, Chapter XI]). Also, it is easy to see that the set of points at which a real-analytic, generic submanifold is not of finite type is a real-analytic subvariety of $M$ (see also [BER4, Sect. 1.5]). Thus, a connected, real-analytic, generic submanifold $M \subset \mathbb{C}^N$ of codimension $d$ is either (a) $\ell$-nondegenerate, for some $\ell$ with $0 \leq \ell \leq N-d$, and of finite type outside a proper real-analytic subvariety of $M$, (b) holomorphically degenerate, or (c) of infinite type at every point (but (b) and (c) are not mutually exclusive).

For a formal generic submanifold $M \subset \mathbb{C}^N$, the notion of (formal) holomorphic nondegeneracy can be defined as follows. We say that a formal $(1,0)$ vector field is *(formally) holomorphic* if its coefficients are independent of $\zeta$. The formal generic submanifold $M$ is said to be *holomorphically nondegenerate* at 0 if there are no nontrivial (formal) holomorphic vector fields tangent to $M$. If $M$ is a real-analytic generic submanifold, then it is (formally) holomorphically nondegenerate at 0 as a formal submanifold if and only if it is holomorphically nondegenerate at 0 as a real-analytic one (i.e. in the sense defined in the introduction). Moreover, if $M$ is a smooth generic submanifold, then it is holomorphically nondegenerate at 0 (as a formal submanifold) if and only there exists a sequence of points $p_j \in M$ tending to 0 such that $M$ is finitely nondegenerate at each $p_j$. The reader is referred to [BER4, Chapter XI] for these results.

The following result shows necessity of the hypotheses in Theorem 2.1.1.

**Theorem 2.2.1.** *Let $M \subset \mathbb{C}^N$ be a formal generic submanifold. Suppose either of the following hold.*

*(i)* *$M$ is holomorphically degenerate at* 0.
*(ii)* *$M$ is weighted homogeneous, i.e. defined by the vanishing of weighted homogeneous polynomials, and of infinite type at every point (as a real-analytic submanifold).*

*Then for any integer $K > 0$ there exist local formal invertible mappings*

$$H^1, H^2 : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$$

*mapping $M$ into itself such that*

$$\frac{\partial^{|\alpha|} H^1}{\partial Z^\alpha}(0) = \frac{\partial^{|\alpha|} H^2}{\partial Z^\alpha}(0), \quad \forall |\alpha| \le K, \tag{2.2.2}$$

*but $H^1 \not\equiv H^2$. If $M$ is real-analytic, then $H_1$ and $H_2$ can be chosen to be biholomorphic near* 0.

In the real-analytic case, Theorem 2.2.1 was proved in [BER2]. The proof of Theorem 2.2.1 in the general case is similar to that in the real-analytic case and the modifications are left to the reader.

## 3. Tools for the proofs

### *3.1. The Segre mappings*

We keep the notation introduced in the previous sections; e.g. $M$ is a formal generic submanifold of codimension $d$ defined by the formal power series $\rho = (\rho_1, \dots, \rho_d)$. Recall that $Z = (z, w)$, with $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_d)$, and $\zeta = (\chi, \tau)$, with $\chi = (\chi_1, \dots, \chi_n)$ and $\tau = (\tau_1, \dots \tau_d)$, are normal coordinates for $M$ at 0, so that $M$ is defined by

$$w_j - Q_j(z, \chi, \tau), \quad j = 1, \dots, d, \tag{3.1.1}$$

where the $Q_j \in \mathbb{C}[[z, \chi, \tau]]$ satisfy

$$Q_j(0, \chi, \tau) \sim Q_j(z, 0, \tau) \sim \tau_j. \tag{3.1.2}$$

As mentioned in Sect. 1, $M$ is also defined by

$$\tau_j - \bar{Q}_j(\chi, z, w), \quad j = 1, \dots, d. \tag{3.1.3}$$

Consider, for each integer $k \ge 1$, the formal mapping $v^k : (\mathbb{C}^{kn}, 0) \to (\mathbb{C}^N, 0)$ defined as follows. For $k = 2j$,

$$\begin{aligned} v^{2j}(z, \chi^1, \dots, z^{j-1}, \chi^j) := \Big(& z, Q(z, \chi^1, \bar{Q}(\chi^1, z^1, \\ & Q(z^1, \chi^2, \dots, \bar{Q}(\chi^{j-1}, z^{j-1}, Q(z^{j-1}, \chi^j, 0)) \dots)))\Big), \end{aligned} \tag{3.1.4}$$

and, for $k = 2j + 1$,

$$\begin{aligned} v^{2j+1}(z, \chi^1, \dots, z^{j-1}, \chi^j, z^j) = \Big(& z, Q(z, \chi^1, \bar{Q}(\chi^1, z^1, \\ & Q(z^1, \chi^2, \dots, Q(z^{j-1}, \chi^j, \bar{Q}(\chi^j, z^j, 0)) \dots)))\Big). \end{aligned} \tag{3.1.5}$$

(In (3.1.4) and (3.1.5), $z^r$ denotes $(z_1^r, \dots, z_n^r)$ and similarly $\chi^r$ denotes $(\chi_1^r, \dots, \chi_n^r)$ for $r = 1, \dots, j$.) For $k = 0$, we set $v^0 = (0, 0)$. We shall refer to the mapping $v^k$ as the $k$th *Segre mapping* of $M$.

**Proposition 3.1.6.** *For any defining series $\rho \in \mathbb{C}[[Z,\zeta]]^d$ of $M$ and any $k \geq 0$,*

$$\rho(v^{k+1}(z,\chi^1,z^1,\ldots),\bar{v}^k(\chi^1,z^1,\ldots)) \sim 0. \tag{3.1.7}$$

*Proof.* For simplicity, we only consider the case where $k = 2j$. It follows from (3.1.4) and (3.1.5) that

(3.1.8)
$$v^{2j+1}(z,\chi^1,\ldots,z^{j-1},\chi^j,z^j) \sim \left(z, Q(z,\bar{v}^{2j}(\chi^1,\ldots,z^{j-1},\chi^j,z^j)\right).$$

It suffices to show (3.1.7) for the defining series given by (3.1.1), for which (3.1.7) is an immediate consequence of (3.1.8). This completes the proof. □

The following characterization of finite type will be important. If $v$ is a formal mapping $(\mathbb{C}^m,0) \to (\mathbb{C}^l,0)$, then we shall write $\mathrm{Rk}(v)$ to denote the rank of the matrix $(\partial v_i/\partial x_j)$, $i = 1,\ldots,l$, $j = 1,\ldots m$, in $\mathbb{A}^l$, where $\mathbb{A}$ denotes the field of fractions of $\mathbb{C}[[x_1,\ldots,x_m]]$. We shall also use the notation $\mathrm{rk}(\partial v_i/\partial x_j)$ for this rank.

**Theorem 3.1.9.** *Let $M$ be a formal generic submanifold of $\mathbb{C}^N$ through 0. Then, the following are equivalent:*

*(i) $M$ is of finite type at 0;*
*(ii) There exists $k_1 \leq d+1$ such that the rank $\mathrm{Rk}(v^k)$ is $N$ for $k \geq k_1$.*

For the proof of Theorem 3.1.9, we shall need special coordinates for a formal generic submanifold. These will be presented in Sect. 3.2 below. The proof of Theorem 3.1.9 will be given in Sect. 3.3.

### *3.2. Formal canonical coordinates*

Recall that $\mathcal{D}_M$ denotes the $\mathbb{C}[[Z,\zeta]]$-module generated by all the formal $(1,0)$ and $(0,1)$ vector fields tangent to the formal generic submanifold $M$ of $\mathbb{C}^N$. We define the integers $m_1,\ldots,m_h$, also called the *Hörmander numbers* of $M$ at 0, as follows. The number $m_1$ is the smallest integer for which there exists a commutator $C$ of vector fields in $\mathcal{D}_M$ of length $m_1$[1] such that $C(0)$ is not in the span of $\mathcal{D}_M(0) \subset T_0'\mathbb{C}^{2N}$. We define the subspace $E_1 \subset T_0'\mathbb{C}^{2N}$ to be the linear span of $\mathcal{D}_M(0)$ and the values at 0 of all commutators of vector fields in $\mathcal{D}_M$ of length $m_1$. We define $l_1$ by

$$l_1 = \dim E_1 - 2(N-d). \tag{3.2.1}$$

We define inductively the numbers $m_1 < m_2 < \ldots < m_h$ and subspaces $E_1 \subset E_2 \subset \ldots \subset E_h = T_0'\mathbb{C}^{2N}$ as follows. The number $m_{k+1}$ is the

[1] The length of a commutator is the number of vector fields in $\mathcal{D}_M$ appearing; e.g., the commutator $[X,[Y,Z]]$, with $X,Y,Z \in \mathcal{D}_M$, has length 3.

smallest integer for which there exists a commutator $C$ of vector fields in $\mathcal{D}_M$ of length $m_{k+1}$ such that $C(0) \notin E_k$. The subspace $E_{k+1}$ is then defined as the span of $E_k$ and the values at 0 of all commutators of vector fields in $\mathcal{D}_M$ of length $m_{k+1}$. We define

$$(3.2.2)\quad l_{k+1} = \dim E_{k+1} - \dim E_k = \dim E_{k+1} - 2(N-d) - \sum_{i=1}^{k} l_i.$$

It is clear that this process terminates after a finite number of steps. We shall call the number $l_j$ the *multiplicity* of the Hörmander number $m_j$. It is also convenient to use the notation $\mu_1, \ldots, \mu_r$ for the Hörmander numbers repeated according to their multiplicities, so that $r = \sum_{j=1}^{h} l_j$.

The following theorem will be used in the proof of Theorem 3.1.9.

**Theorem 3.2.3.** *Let $M$ be a formal generic submanifold of $\mathbb{C}^N$ of codimension $d$ through 0. Let $2 \le \mu_1 \le \ldots \le \mu_r$ be the Hörmander numbers of $M$ at 0 repeated according to their multiplicities. There exists a formal change of coordinates $Z = Z(z, w', w'')$, with $z = (z_1, \ldots, z_n)$, $w' = (w'_1, \ldots, w'_r)$, $w'' = (w''_1, \ldots, w''_{d-r})$, $N = n + d$, satisfying the following. The defining series $\rho$ of $M$, after the formal change of coordinates $(Z, \zeta) = (Z(z, w', w''), \bar{Z}(\chi, \tau', \tau''))$, satisfies*

(3.2.4)

$$\rho(z, w', w'', \chi, \tau', \tau'') = a(z, w', w'', \chi, \tau', \tau'') \begin{pmatrix} w' - Q'(z, \chi, \tau', \tau'') \\ w'' - Q''(z, \chi, \tau', \tau'') \end{pmatrix}$$

*where $a(z, w', w'', \chi, \tau', \tau'')$ is a $d \times d$ matrix of formal power series which is invertible at 0, and $Q'(z, \chi, \tau', \tau'')$ and $Q''(z, \chi, \tau', \tau'')$ are of the form*

(3.2.5)

$$Q'_k(z, \chi, \tau', \tau'') \sim \tau'_k + p_k(z, \chi, \tau'_1, \ldots, \tau'_{k-1}) + A_k(z, \chi, \tau', \tau'')\tau'' + R_k(z, \chi, \tau')$$

$$Q''(z, \chi, \tau', \tau'') \sim \tau'' + B(z, \chi, \tau', \tau'')\tau'',$$

*where $k = 1, \ldots, r$. Here, $p_k(z, \chi, \tau'_1, \ldots, \tau'_{k-1})$ is a weighted homogeneous polynomial of degree $\mu_k$, where $z$ and $\chi$ have weight one and $\tau'_j$ has weight $\mu_j$ for $j = 1, \ldots r$, $R_k(z, \chi, \tau')$ is a formal power series which is $O(\mu_k + 1)$ (i.e. involving only terms which are weighted homogeneous of degree at least $\mu_k + 1$), $A_k(z, \chi, \tau', \tau'')$ and $B(z, \chi, \tau', \tau'')$ are matrices of formal power series without constant terms. Moreover, we have*

$$(3.2.6)\quad \begin{aligned} Q'(z, 0, \tau', \tau'') &\sim Q'(0, \chi, \tau', \tau'') \sim \tau', \\ Q''(z, 0, \tau', \tau'') &\sim Q''(0, \chi, \tau', \tau'') \sim \tau''. \end{aligned}$$

The proof of Theorem 3.2.3 can be extracted from the proof of [BER4, Theorem 4.5.1]. The reader should observe that $M$ is of finite type at 0 if and only if $r = d$. In this case, there are no $w''$ variables in Theorem 3.2.3, i.e. $w = w'$ and $\tau = \tau'$ in (3.2.4–6).

*3.3. Proof of Theorem 3.1.9*

In view of Theorem 3.2.3, we may assume that we have formal coordinates $Z = (z, w', w'')$, $\zeta = (\chi, \tau', \tau'')$, as described in Theorem 3.2.3, such that $M$ is defined by $w' - Q'(z, \chi, \tau', \tau'')$ and $w'' - Q''(z, \chi, \tau', \tau'')$, where $Q'$ and $Q''$ satisfy (3.2.5) and (3.2.6). Let us write

$$Q'(z, \chi, \tau', \tau'') \sim \tau' + p(z, \chi, \tau') + R(z, \chi, \tau, \tau''), \tag{3.3.1}$$

where

$$p(z, \chi, \tau') = (p_1(z, \chi), \ldots, p_r(z, \chi, \tau'_1, \ldots, \tau'_{r-1})) \tag{3.3.2}$$

are weighted homogeneous polynomials as in Theorem 3.2.3 and $R = (R_1, \ldots R_r)$ are the remainder terms of higher (weighted) homogeneity. Consider the homogeneous generic submanifold $M^0$ of $\mathbb{C}^N$ given by

$$w' = \bar{w}' + p(z, \bar{z}, \bar{w}'), \quad w'' = \bar{w}''. \tag{3.3.3}$$

Observe that $M^0$ has the same Hörmander numbers as $M$ (with multiplicity). For each fixed $k$, we denote, for simplicity of notation, the variables in the space $\mathbb{C}^{kn}$, where the $k$th Segre mappings are defined, by $(z, \xi)$, where $\xi \in \mathbb{C}^{(k-1)n}$. We denote by $v^k(z, \xi)$ the $k$th Segre mapping of $M$ at 0 as defined by (3.1.4) and (3.1.5), and by $v_0^k(z, \xi)$ the $k$th Segre mapping of the formal generic submanifold associated to $M^0$ at 0. The $j$th component of the mapping $v_0^k$ is a homogeneous polynomial (in the usual sense; i.e. all components of $z$ and $\xi$ have weight one) of degree $\mu_j$, where $\mu_j$ denotes the $j$th Hörmander number (with multiplicity) of $M^0$ (or $M$) at 0. The $z$ component of the mappings $v^k(z, \xi)$ and $v_0^k(z, \xi)$ coincide, and are equal to $z$. Moreover, the $w''$ components also coincide, and are equal to 0. Let us separate the $z$, $w'$, and $w''$ components of the mappings $v^k$ and $v_0^k$ and write

$$v^k(z, \xi) = (z, g^k(z, \xi), 0), \quad v_0^k(z, \xi) = (z, g_0^k(z, \xi), 0), \tag{3.3.4}$$

where $(g_0^k)_j(z, \xi)$ is a homogeneous polynomial of degree $\mu_j$. We have, for $j = 1, \ldots, r$,

$$g_j^k(z, \xi) \sim (g_0^k)_j(z, \xi) + O(\mu_j + 1), \tag{3.3.5}$$

where $O(\nu + 1)$ denotes a power series consisting only of terms of degrees higher than $\nu$.

It follows from [BER1, Proposition 2.4.1] (see also [BER4, Proposition 10.6.27]) that there exists $k_1$, with $1 \le k_1 \le d+1$, such that $\mathrm{Rk}(v_0^k) = n+r$ for $k \ge k_1$. Observe that the determinant of an $m \times m$ matrix $A$ of power series, where the $j$th row of $A$ is of the form $f_j = f_j^0 + O(d_j + 1)$ for some homogeneous polynomial $f_j^0$ of degree $d_j$, is of the form

$$\det A = \det A^0 + O(d_1 + \ldots + d_m + 1), \tag{3.3.6}$$

where $A^0$ is the matrix with rows $f_1^0, \dots f_m^0$. It follows from this observation and (3.3.5) that $\mathrm{Rk}(v^k) \geq \mathrm{Rk}(v_0^k)$, and hence $\mathrm{Rk}(v^k) \geq n + r$ for $k \geq k_1$. On the other hand, by the form (3.3.4) of $v^k$, we have $\mathrm{Rk}(v^k) \leq n + r$ for any $k \geq 1$. Thus,

$$\mathrm{Rk}(v^k) = n + r, \quad \forall k \geq k_1, \tag{3.3.7}$$

The equivalence of (i) and (ii) of Theorem 3.1.9 follows from the fact that $M$ is of finite type at 0 if and only if $r = d$, i.e.if and only if $\mathrm{Rk}(v^k) = n+d = N$, for $k \geq k_1$. □

### 3.4. Basic identity for formal mappings

An important tool, in combination with the Segre mappings, in the proofs of the theorems in Sect. 2.1 will be the basic identity which we shall present in this section. We keep the notation established in the previous sections. In what follows, $M$ and $M'$ denote fixed formal generic submanifolds of codimension $d$ and $d'$ through the origin of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively.

We choose normal coordinates $Z = (z, w)$, $\zeta = (\chi, \tau)$ for $M$ as in Sect. 1, and, similarly, normal coordinates $Z' = (z', w')$, $\zeta' = (\chi', \tau')$ for $M'$. There is an associated coordinate system on $J^l(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)} \cong \mathbb{C}^{K(l)}$, where $K(l)$ denotes the dimension of this jet space. We shall use a scaled coordinate system whose coordinates we shall denote by

$$\Lambda = (\lambda_{z^\alpha w^\beta}, \mu_{z^\gamma w^\delta})_{1 \leq |\alpha|+|\beta|, |\gamma|+|\delta| \leq l} \, . \tag{3.4.1}$$

For a formal mapping $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$, write $H(z, w) = (F(z, w), G(z, w))$, where $F = (F_1, \dots, F_{n'})$ and $G = (G_1, \dots G_{d'})$. In the scaled coordinates (3.4.1) we have, for any non-negative integer $l$,

$$j_0^l(H) = (\lambda_{z^\alpha w^\beta}, \mu_{z^\gamma w^\delta})_{1 \leq |\alpha|+|\beta|, |\gamma|+|\delta| \leq l} \, , \tag{3.4.2}$$

where

$$\lambda_{z^\alpha w^\beta} = \frac{\partial^{|\alpha|+|\beta|} F}{\partial z^\alpha \partial w^\beta}(0, 0), \quad \mu_{z^\gamma w^\delta} = \frac{\partial^{|\gamma|+|\delta|} G}{\partial z^\gamma \partial w^\delta}(0, 0). \tag{3.4.3}$$

For each fixed $l$, we shall split, and reorder, the variables $\Lambda$ in (3.4.1) as follows

$$\Lambda = (\Lambda', \Lambda'') \, , \tag{3.4.4}$$

where

$$\Lambda'' = (\mu_{z^\gamma})_{1 \leq |\gamma| \leq l} \, , \tag{3.4.5}$$

and the components of $\Lambda'$ are the remaining variables in (3.4.1). We shall denote the number of components of $\Lambda'$ by $K' = K'(l)$ and that of $\Lambda''$ by $K'' = K''(l)$, so that $J^l(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)} \cong \mathbb{C}^{K'} \times \mathbb{C}^{K''}$. We are now in a position to state the basic identity.

**Theorem 3.4.6.** *Let $M$ and $M'$ be formal generic submanifolds through the origin in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively. Assume that $M'$ is $\ell_0$-nondegenerate at 0, and that $n \geq n'$, where $n = N - d$, $n' = N' - d'$, $d = \operatorname{codim} M$, and $d' = \operatorname{codim} M'$. Then for each $\tilde{\jmath} = (j_1, \ldots, j_{n'})$, with $1 \leq j_1 < \ldots < j_{n'} \leq n$, and for every $\alpha \in \mathbb{Z}_+^N$, there exists a formal power series mapping of the form*

$$(3.4.7)\quad \Psi_\alpha^{\tilde{\jmath}}(Z,\zeta,\zeta',\Lambda) \sim \sum_{\beta,\gamma,\delta,\kappa} \frac{d_{\beta\gamma\delta\kappa}(\Lambda')}{(\det(\lambda_{z_{j_p}}^l)_{1\leq l,p\leq n'})^{l_{\beta\gamma\delta\kappa}}} Z^\beta \zeta^\gamma \zeta'^\delta \Lambda''^\kappa,$$

*where $\Lambda = (\Lambda', \Lambda'') \in \mathbb{C}^{K'} \times \mathbb{C}^{K''}$ with $K' = K'(\ell_0 + |\alpha|)$ and $K'' = K''(\ell_0 + |\alpha|)$, $d_{\beta\gamma\delta\mu}(\Lambda')$ are $\mathbb{C}^{N'}$ valued polynomials in $\mathbb{C}^{K'}$, and $l_{\beta\gamma\delta\mu}$ nonnegative integers, satisfying the following. For every formal mapping $H \in \hat{\mathcal{F}}(M, M')$ there exists $\tilde{\jmath} = (j_1, \ldots, j_{n'})$ such that*

$$(3.4.8)\qquad \det\left(\frac{\partial F_l}{\partial z_{j_p}}(0)\right)_{1\leq l,p\leq n'} \neq 0$$

*and, for all $\alpha \in \mathbb{Z}_+^N$,*

$$(3.4.9)\qquad \partial^\alpha H(Z) - \Psi_\alpha^{\tilde{\jmath}}\left(Z,\zeta,\bar{H}(\zeta),(\partial^\beta \bar{H}(\zeta))_{1\leq|\beta|\leq\ell_0+|\alpha|}\right) \sim a(Z,\zeta)\rho(Z,\zeta),$$

*where $a(Z,\zeta)$ is a $d\times d$ matrix of formal power series and $\rho = (\rho_1, \ldots, \rho_d)$ is a defining series for $M$. Moreover,* (3.4.9) *holds for any $H \in \hat{\mathcal{F}}(M, M')$ and any $\tilde{\jmath}$ such that* (3.4.8) *holds.*

*If $M$ and $M'$ are real-analytic, then, for any $\tilde{\jmath}$ as above, $\alpha \in \mathbb{Z}_+^N$, and any $\Lambda_0' \in \mathbb{C}^{K'(\ell_0+|\alpha|)}$ satisfying*

$$(3.4.10)\qquad \det\left((\lambda_0)_{z_{j_p}}^l\right)_{1\leq l,p\leq n'} \neq 0,$$

*the series $\Psi_\alpha^{\tilde{\jmath}}(Z,\zeta,\zeta',\Lambda)$ given by* (3.4.7) *defines a holomorphic mapping near the point $(Z,\zeta,\zeta',\Lambda',\Lambda'') = (0,0,0,\Lambda_0',0)$.*

*Remark* 3.4.11. The reader should observe that in substituting the formal mapping $(\partial^\beta \bar{H}(\zeta))_{1\leq|\beta|\leq\ell_0+|\alpha|}$ for $\Lambda$ in (3.4.7), we replace $\Lambda''$ by $(\partial_\chi^\beta \bar{G}(\chi,\tau))_{1\leq|\beta|\leq\ell_0+|\alpha|}$ and $\Lambda'$ by the remaining derivatives. This substitution of formal power series makes sense since, as remarked in Sect. 1, $G(z,0) \sim 0$, and the dependence on $\Lambda'$ is rational. In what follows, we shall, for fixed $l$, decompose, and reorder the components of $(\partial^\beta H(Z)) = (\partial^\beta H(Z))_{1\leq|\beta|\leq l}$ as $((\partial^\beta H(Z))', (\partial^\beta H(Z))'')$, where $(\partial^\beta H(Z))'' = (\partial_z^\beta G(z,w))_{1\leq|\beta|\leq l}$ and $(\partial^\beta H(Z))'$ denotes the remaining derivatives.

*Proof of Theorem* 3.4.6. Recall that, in the chosen normal coordinates, $H$ maps $(M,0) \to (M',0)$ if and only if (1.14) holds. The mapping $H$ belongs to $\hat{\mathcal{F}}(M,M')$ if it also satisfies (1.19), which, as noted in the proof of Corollary 1.27, is equivalent to $\partial F/\partial z(0,0)$ having rank $n'$. Thus, there exists $\tilde{\jmath}$ as in the statement of the theorem such that (3.4.8) holds.

Let $\tilde{\jmath}$, as in the theorem, be given. We shall consider only those mappings $H : (M,0) \to (M',0)$ for which (3.4.8) holds. After renumbering the variables if necessary, we may assume that $\tilde{\jmath} = (1,2,\ldots,n')$.

We take as a basis for the $(0,1)$ vector fields tangent to $M$ (modulo those whose coefficients are in the ideal generated by a set of defining power series of $M$)

$$L_j = \frac{\partial}{\partial \chi_j} + \sum_{k=1}^{d} \bar{Q}_{k,\chi_j}(\chi,z,w)\frac{\partial}{\partial \tau_k}, \quad j = 1,\ldots,n. \tag{3.4.12}$$

In what follows, we shall only use the vector fields $L_1,\ldots,L_{n'}$. We shall also need the following vector fields tangent to $M$ and given by

$$\begin{aligned} \tilde{L}_j &= \frac{\partial}{\partial z_j} + \sum_{k=1}^{d} Q_{k,z_j}(z,\chi,\tau)\frac{\partial}{\partial w_k}, \qquad j = 1,\ldots,n, \\ T_j &= \frac{\partial}{\partial w_j} + \sum_{k=1}^{d} \bar{Q}_{k,w_j}(\chi,z,w)\frac{\partial}{\partial \tau_k}, \qquad j = 1,\ldots,d, \\ V_j &= \tilde{L}_j - \sum_{k=1}^{d} Q_{k,z_j}(z,\chi,\tau)T_k, \qquad j = 1,\ldots,n. \end{aligned} \tag{3.4.13}$$

Note that the $\tilde{L}_j$ form a basis (modulo $(\rho)$) for the $(1,0)$ vector fields tangent to $M$. After applying the $L_j$, for $j = 1,\ldots,n'$, to the second set of equations in (1.14) $|\alpha|$ times, and applying Cramer's rule after each application, we obtain, for $w = Q(z,\chi,\tau)$ or $\tau = \bar{Q}(\chi,z,w)$, any multi-index $\alpha$, and $l = 1,\ldots,d'$,

$$\begin{aligned} &\bar{Q}'_{l,\chi'^\alpha}(\bar{F}(\chi,\tau),F(z,w),G(z,w)) = \\ &\sum_{1\le|\beta|\le|\alpha|} (L^\beta \bar{G}_l(\chi,\tau))P_{\alpha,\beta}\left((L^\gamma \bar{F}(\chi,\tau))_{1\le|\gamma|\le|\alpha|}\right)/\Delta^{2|\alpha|-1}, \end{aligned} \tag{3.4.14}$$

where $\Delta = \Delta(z,w,\chi,\tau) = \det[L_j\bar{F}_k(\chi,\tau)]_{1\le j,k\le n'}$, and $P_{\alpha,\beta}$ are universal polynomials, i.e. independent of $M$, $M'$, and $H$. Note that for any formal power series $h(\chi,\tau)$ and any multi-index $\beta$ we have $L^\beta h(0) = \partial_{\chi^\beta}h(0)$. Thus, $\Delta(0) \neq 0$ by (3.4.8). Also, $L^\beta G(0) = 0$ by the normality of the coordinates, as is easily verified from (1.14). Hence the right hand side of (3.4.14) vanishes at the origin.

By the assumption that $M'$ is $\ell_0$-nondegenerate at 0, there exist $n'$ multi-indices $\alpha^1, \dots, \alpha^{n'}$, with $1 \le |\alpha^j| \le \ell_0$, and $n'$ integers $l_1, \dots, l_{n'} \in \{1, \dots, d'\}$ such that $\det \left[ \bar{Q}'_{l_j, \chi'^{\alpha^j} z'_k}(0) \right] \ne 0$. (See [BER1] or [BER4, Corollary 11.2.14].) Hence, by the implicit function theorem, there exists a unique $\mathbb{C}^{n'}$ valued formal power series $S(\chi', \tau', r)$ with $r = (r_1, \dots, r_{n'})$, so that $S(0,0,0) = 0$ and $X = S(\chi', \tau', r)$ solves the system of equations

$$\bar{Q}'_{l_j, \chi'^{\alpha^j}}(\chi', X, Q'(X, \chi', \tau')) \sim r_j, \quad j = 1, \dots, n'. \tag{3.4.15}$$

For any positive integer $k$, we shall introduce the vector valued variables $(a_\gamma)_{|\gamma| \le k}$, $(b_\beta)_{|\beta| \le k}$, where $\beta, \gamma \in \mathbb{Z}_+^{n'}$, corresponding to $(L^\gamma \bar{F}(\chi, \tau))_{|\gamma| \le k}$, $(L^\beta \bar{G}(\chi, \tau))_{|\beta| \le k}$, respectively. Here $a_\gamma = (a_\gamma^m)_{1 \le m \le n'}$ and $b_\beta = (b_\beta^j)_{1 \le j \le d'}$. We write $(a_k^m)_{1 \le k, m \le n'}$ for $(a_\gamma^m)_{1 \le m \le n', |\gamma| = 1}$. We define

$$R_\alpha \left( (a_\gamma)_{1 \le |\gamma| \le |\alpha|}, (b_\beta^1)_{1 \le |\beta| \le |\alpha|} \right) = \frac{\sum_{1 \le |\beta| \le |\alpha|} (b_\beta^1) P_{\alpha, \beta} \left( (a_\gamma)_{1 \le |\gamma| \le |\alpha|} \right)}{\left( \det(a_k^m)_{1 \le k, m \le n'} \right)^{2|\alpha| - 1}}. \tag{3.4.16}$$

Observe that $R_\alpha$ is a universal rational function that vanishes when $b_\beta^1 = 0$, $|\beta| \le |\alpha|$, and whose denominator is a power of $\det(a_k^m)_{1 \le k, m \le n'}$. It follows from the above that, for $w = Q(z, \chi, \tau)$ or $\tau = \bar{Q}(\chi, z, w)$, we have the identity

$$F(z, w) = \Theta \left( (L^\gamma \bar{F}(\chi, \tau))_{|\gamma| \le \ell_0}, (L^\beta \bar{G}(\chi, \tau))_{|\beta| \le \ell_0} \right), \tag{3.4.17}$$

where

$$\begin{aligned} &\Theta((a_\gamma)_{|\gamma| \le \ell_0}, (b_\beta)_{|\beta| \le \ell_0}) = \\ &S \left( a_0, b_0, \left( R_{\alpha^j} \left( (a_\gamma)_{1 \le |\gamma| \le |\alpha^j|}, (b_\beta^{l_j})_{1 \le |\beta| \le |\alpha^j|} \right) \right)_{1 \le j \le n'} \right). \end{aligned} \tag{3.4.18}$$

Now, since $F(z, w)$ is a power series in $(z, w)$ only, we have, for any multi-index $\nu = (\nu', \nu'')$,

$$V^{\nu'} T^{\nu''} F(z, w) = \frac{\partial^{|\nu|} F}{\partial z^{\nu'} \partial w^{\nu''}}(z, w). \tag{3.4.19}$$

By applying $V^{\nu'} T^{\nu''}$ to the identity (3.4.17), we obtain

$$\begin{aligned} \frac{\partial^{|\nu|} F}{\partial z^{\nu'} \partial w^{\nu''}}(z, w) = \Theta_\nu \Big( &(V^{\delta'} T^{\delta''} L^\gamma \bar{F}(\chi, \tau))_{|\delta| + |\gamma| \le \ell_0 + |\nu|}, \\ &(V^{\kappa'} T^{\kappa''} L^\beta \bar{G}(\chi, \tau))_{|\kappa| + |\beta| \le \ell_0 + |\nu|} \Big), \end{aligned} \tag{3.4.20}$$

where we have used the notation $\delta = (\delta', \delta'')$ and $\kappa = (\kappa', \kappa'')$. The identity (3.4.20) holds when $w = Q(z, \chi, \tau)$ or $\tau = \bar{Q}(\chi, z, w)$. Observe that the power series $\Theta_\nu$ depends only on $\Theta$ and its derivatives.

By substituting (3.4.12) and (3.4.13) in (3.4.20), we obtain for $w = Q(z, \chi, \tau)$ or $\tau = \bar{Q}(\chi, z, w)$,

$$\frac{\partial^{|\nu|} F}{\partial z^{\nu'} \partial w^{\nu''}}(z, w) =$$

$$\Phi^1_\nu(z, w, \chi, \tau, \bar{F}(\chi, \tau), \bar{G}(\chi, \tau), (\partial^\alpha \bar{H}(\chi, \tau))', (\partial^\alpha \bar{H}(\chi, \tau))''). \tag{3.4.21}$$

where $(\partial^\alpha \bar{H}(\chi, \tau)) = (\partial^\alpha \bar{H}(\chi, \tau))_{|\alpha| \le \ell_0 + |\nu|}$ and we use the notation $(\partial^\alpha \bar{H}(\chi, \tau)) = ((\partial^\alpha \bar{H}(\chi, \tau))', (\partial^\alpha \bar{H}(\chi, \tau))'')$ as explained in Remark 3.4.11. Observe that the power series $\Phi^1_\nu$ depends only on $M$ and $M'$, and not on the mapping $H$. Using the notation

$$\bar{\Lambda} = ((\lambda_{\chi^\alpha \tau^\beta})_{1 \le |\alpha| + |\beta| \le \ell_0 + |\nu|}, (\mu_{\chi^\gamma \tau^\delta})_{1 \le |\gamma| + |\delta| \le \ell_0 + |\nu|}), \tag{3.4.22}$$

decomposing $\bar{\Lambda}$, and reordering its components in an analogous fashion as $\bar{\Lambda} = (\bar{\Lambda}', \bar{\Lambda}'')$ with $\bar{\Lambda}'' = (\mu_{\chi^\beta})_{1 \le |\beta| \le \ell_0 + |\nu|}$, it follows from (3.4.18) and (3.4.20) that the power series

$$\Phi^1_\nu\left(z, w, \chi, \tau, \chi', \tau', \bar{\Lambda}', \bar{\Lambda}''\right) \tag{3.4.23}$$

is of the form
(3.4.24)

$$\Phi^1_\nu(z, w, \chi, \tau, \chi', \tau', \bar{\Lambda}', \bar{\Lambda}'') \sim \sum_{\beta, \gamma, \delta, \kappa} \frac{e_{\beta\gamma\delta\kappa}(\bar{\Lambda}')}{\det(\lambda^j_{\chi_k})^{l_{\beta\gamma\delta\kappa}}} Z^\beta \zeta^\gamma \zeta'^\beta \bar{\Lambda}''^\kappa ,$$

where $e_{\beta\gamma\delta\kappa}(\bar{\Lambda})$ are $\mathbb{C}^{n'}$ valued polynomials and $l_{\beta\gamma\delta\kappa}$ nonnegative integers. We have used here the fact that

$$R_{\alpha^j}\left((a_\gamma)_{1 \le |\gamma| \le |\alpha^j|}, (b^{l_j}_\beta)_{1 \le |\beta| \le |\alpha^j|}\right) = 0, \tag{3.4.25}$$

when $(b^{l_j}_\beta)_{1 \le |\beta| \le |\alpha^j|} = 0$. In view of (3.4.21) and (3.4.24), we can take the first $n'$ components of $\Psi^{\tilde{j}}_\nu(Z, \zeta, \zeta', \Lambda)$, with the fixed choice of $\tilde{j}$ above, in the conclusion of the theorem to be $\Phi^1_\nu(z, w, \chi, \tau, \chi', \tau', \Lambda', \Lambda'')$.

To complete the construction of $\Psi^{\tilde{j}}_\nu$, we need to find the components corresponding to $G$ and its derivatives. For this we substitute (3.4.21) with $\nu = 0$ in the first set of equations in (1.14), and apply the vector fields $V_j$ and $T_j$ to the identity thus obtained, as above. We obtain

$$\frac{\partial^{|\nu|} G}{\partial z^{\nu'} \partial w^{\nu''}}(z, w) =$$

$$\Phi^2_\nu(z, w, \chi, \tau, \bar{F}(\chi, \tau), \bar{G}(\chi, \tau), (\partial^\alpha \bar{H}(\chi, \tau))', (\partial^\alpha \bar{H}(\chi, \tau))'') \tag{3.4.26}$$

where $(\partial^\alpha \bar{H}(\chi,\tau)) = (\partial^\alpha \bar{H}(\chi,\tau))_{|\alpha|\le \ell_0+|\nu|}$ and we use the notation $(\partial^\alpha \bar{H}(\chi,\tau)) = ((\partial^\alpha \bar{H}(\chi,\tau))', (\partial^\alpha \bar{H}(\chi,\tau))'')$ as above; (3.4.26) holds for $w = Q(z,\chi,\tau)$ or $\tau = \bar{Q}(\chi,z,w)$. We omit the details of this construction, since it is similar to the one above for the component $F$. Note, by inspecting the construction above, that the function $\Psi_\nu^{\bar{j}}$ is defined only in terms of the defining equations of $M$ and $M'$, and does not depend on the existence or choice of a mapping $H$. The proof of the formal part of the theorem is complete.

Suppose that $M$ and $M'$ are also real-analytic. Then the function $S(\chi',\tau',\tau)$, defined by (3.4.15), is holomorphic in a neighborhood of the origin. The fact, noted above, that each rational function $R_\alpha((a_\gamma),(b_\beta^l))$ vanishes when $(b_\beta^l) = 0$ implies that the functions $\Psi_\alpha^{\bar{j}}(Z,\zeta,\zeta',\Lambda)$ above are holomorphic in a neighborhood of $(Z,\zeta,\zeta';\Lambda',\Lambda'') = (0,0,0,\Lambda_0',0)$ for any $\Lambda_0'$ such that (3.4.10) holds. This completes the proof of Theorem 3.4.6. □

Although Theorem 2.1.1 is a consequence of Theorem 2.1.5, we conclude this section by giving a direct proof of it.

*Proof of Theorem* 2.1.1. We take normal coordinates for $M$ and $M'$ as in the proof of Theorem 3.4.6. Let $\bar{j} = (j_1,\ldots,j_{n'})$ be such that $\det(\partial F_l^m/\partial z_{j_p}(0,0))_{1\le l,p\le n'} \ne 0$, for $m = 1,2$. By Proposition 3.1.6 and the basic identity, Theorem 3.4.6, it follows that

$$\begin{aligned} &\partial^\alpha H(Z) \sim \Psi_\alpha^{\bar{j}}\left(Z,\zeta,\bar{H}(\zeta),(\partial^\beta \bar{H}(\zeta))_{1\le|\beta|\le \ell_0+|\alpha|}\right), \\ &\text{for } Z = v^{k+1}(z,\chi^1,z^1,\ldots),\ \zeta = \bar{v}^k(\chi^1,z^1,\ldots), \end{aligned} \tag{3.4.27}$$

for any $k \ge 0$, where $v^l$ denotes the Segre mapping defined in §3.1 and $v^0 = (0,0)$. Hence, if for some $k_0$ one has $j_0^{\ell_0 k_0}(H^1) = j_0^{\ell_0 k_0}(H^2)$, then it follows from (3.4.27), for any $k \le k_0$, that

$$(\partial^\alpha H^1)\circ v^k \sim (\partial^\alpha H^2)\circ v^k,\ \ \forall \alpha : |\alpha| \le \ell_0(k_0-k), \tag{3.4.28}$$

as can be seen by an induction on $k$. In particular, we have

$$(H^1 - H^2)\circ v^{k_0} \sim 0. \tag{3.4.29}$$

By Theorem 3.1.9, there exists $k_1$, with $k_1 \le d+1$, such that $\mathrm{Rk}(v^k) = N$, for $k \ge k_1$. It then follows from standard facts about formal power series (see e.g. [BER4, Proposition 5.3.5]) that (3.4.29) implies $H^1 \sim H^2$ if $k_0 \ge k_1$. The proof of Theorem 2.1.1 is complete. □

## 4. Proofs of the main results

### *4.1. Proof of Theorem 2.1.5*

It suffices to prove Theorem 2.1.5 in normal coordinates. Thus, we take normal coordinates for $M$ and $M'$ as in previous sections. We also keep the notation introduced in the beginning of §3.4 and in Remark 3.4.11. Consider the linear mapping $D_k : \mathbb{C}^{kn} \to \mathbb{C}^{2kn}$ defined as follows. For $k = 2j, j \geq 1$, we set

$$\begin{aligned} &D_{2j}(\chi^1, z^1, \dots, z^{j-1}, \chi^j, z^j) := \\ (4.1.1) \qquad &(0, \chi^1, z^1, \dots, z^{j-1}, \chi^j, z^j, \chi^j, z^{j-1}, \dots, z^1, \chi^1), \end{aligned}$$

and for $k = 2j - 1, j \geq 1$, we set

$$\begin{aligned} &D_{2j-1}(\chi^1, z^1, \dots, z^{j-1}, \chi^j) := \\ (4.1.2) \qquad &(0, \chi^1, z^1, \dots, z^{j-1}, \chi^j, z^{j-1}, \dots, z^1, \chi^1). \end{aligned}$$

We remind the reader that $v^k$ denotes the $k$th Segre mapping as defined by (3.1.4) and (3.1.5). We shall need the following.

**Lemma 4.1.3.** *For any $k \geq 1$, the following hold.*

$$v^{2k} \circ D_k \sim 0. \tag{4.1.4}$$

*For $k = 2j$, $j \geq 1$,*

$$\begin{aligned} (4.1.5) \qquad \operatorname{rk} \Bigg( &\frac{\partial v^{2k}}{\partial z} \circ D_k, \frac{\partial v^{2k}}{\partial \chi^{j+1}} \circ D_k, \frac{\partial v^{2k}}{\partial z^{j+1}} \circ D_k, \\ &\dots, \frac{\partial v^{2k}}{\partial \chi^k} \circ D_k \Bigg) = \operatorname{Rk}(v^k). \end{aligned}$$

*and for $k = 2j - 1$, $j \geq 1$,*

$$\begin{aligned} (4.1.6) \quad \operatorname{rk} \Bigg( &\frac{\partial v^{2k}}{\partial z} \circ D_k, \frac{\partial v^{2k}}{\partial z^j} \circ D_k, \frac{\partial v^{2k}}{\partial \chi^{j+1}} \circ D_k, \frac{\partial v^{2k}}{\partial z^{j+1}} \circ D_k, \\ &\dots, \frac{\partial v^{2k}}{\partial \chi^k} \circ D_k \Bigg) = \operatorname{Rk}(v^k). \end{aligned}$$

*In particular,*

$$\operatorname{rk} \left( \frac{\partial v^{2k}}{\partial z} \circ D_k, \frac{\partial v^{2k}}{\partial \chi^1} \circ D_k, \frac{\partial v^{2k}}{\partial z^1} \circ D_k, \dots, \frac{\partial v^{2k}}{\partial \chi^k} \circ D_k \right) \geq \operatorname{Rk}(v^k). \tag{4.1.7}$$

*Proof.* Property (4.1.4) follows by making repeated use of the identities,

$$Q(z,\chi,\bar{Q}(\chi,z,w))\sim w,\quad \bar{Q}(\chi,z,Q(z,\chi,\tau))\sim\tau, \tag{4.1.8}$$

which are easily checked (see also [BER4, Remark 4.2.30]).

To prove (4.1.5) and (4.1.6), we first write

$$v^l(z,\chi^1,z^1\dots,)=(z,u^l(z,\chi^1,z^1\dots)), \tag{4.1.9}$$

where $u^l=(u^l_1,\dots,u^l_d)$. We also write $(z,\xi^{(l)})$ for the variables $(z,\chi^1,z^1\dots)\in\mathbb{C}^{ln}$. Observe, by the form of $v^l$ given by (3.1.4) and (3.1.5), that

$$\operatorname{rk}\left(\frac{\partial v^l(z,\xi^{(l)})}{\partial(z,\xi^{(l)})}\right)=n+\operatorname{rk}\left(\frac{\partial u^l(z,\xi^{(l)})}{\partial\xi^{(l)}}\right). \tag{4.1.10}$$

We shall complete the proof of Lemma 4.1.3 in the case where $k=2j$, and leave the odd case to the reader. Thus, we shall prove (4.1.5). We have

$$\begin{aligned} &u^{4j}(z,\chi^1,z^1,\dots,\chi^{2j})=\\ &\quad Q(z,\chi,\bar{Q}(\chi,z^1,\dots\bar{Q}(\chi^j,z^j,u^{2j}(z^j,\chi^{j+1}\dots,z^{2j-1},\chi^{2j}))\dots)). \end{aligned} \tag{4.1.11}$$

For fixed $k=2j$, we shall write $\xi^{(2k)}=(\xi',z^j,\xi'')$, where $\xi'=(\chi^1,z^1,\dots,\chi^j)$ and $\xi''=(\chi^{j+1},z^{j+1},\dots,\chi^{2j})$. We claim that

$$\operatorname{rk}\left(\frac{\partial u^{2k}(z,\xi',z^j,\xi'')}{\partial\xi''}\circ D_k\right)=\operatorname{rk}\left(\frac{\partial u^k(z^j,\xi'')}{\partial\xi''}\right). \tag{4.1.12}$$

Since $D_k(\xi',z^j)=(0,\xi',z^j,\tilde{\xi}')$, where $\tilde{\xi}'=(\chi^j,z^{j-1},\dots,\chi^1)$, (4.1.12) follows from the chain rule, by using (4.1.11) and the fact that

$$\frac{\partial Q}{\partial\tau}(0,0,0)=\frac{\partial\bar{Q}}{\partial w}(0,0,0)=I_{d\times d}, \tag{4.1.13}$$

where $I_{d\times d}$ denotes the $d\times d$ identity matrix. (The identity (4.1.13) is a consequence of (3.1.2).) The desired equality (4.1.5) is an easy consequence of (4.1.12). This completes the proof of Lemma 4.1.3. □

We now return to the proof of Theorem 2.1.5. We fix $\tilde{\jmath}$ as in that theorem. Let $k_1$ be the integer provided by Theorem 3.1.9. We shall use the notation $(z,\xi)$ for $(z,\chi^1,z^1,\dots)\in\mathbb{C}^{2k_1n}$ as in the proof of Lemma 4.1.3 above. We claim that there exist rational functions $\sigma_0((\Lambda')^{(0)}),\dots,\sigma_{2k_1-1}((\Lambda')^{(2k_1-1)})$, where $(\Lambda')^{(k)}\in\mathbb{C}^{K'(k\ell_0+1)}$, with the following property. (In what follows, we shall consider each $\sigma_k$ as a function of $\Lambda'\in\mathbb{C}^{K'(2k_1\ell_0)}$ by letting it be independent of those components of $\Lambda'$ that correspond to jets of order higher than $k\ell_0+1$ under the identification given at the beginning of Sect. 3.4.)

There exists a $\mathbb{C}^{N'}$ valued formal power series in $(z,\xi)$, $\Xi^{\tilde{\jmath}}(z,\xi,\Lambda')$, of the form

(4.1.14)
$$\Xi^{\tilde{\jmath}}(z,\xi,\Lambda') \sim \sum_{\gamma,\delta} \frac{d_{\gamma\delta}(\Lambda')}{\sigma_{2k_1-1}(\Lambda')^{l^{2k_1-1}_{\gamma\delta}}\,\sigma_{2k_1-2}(\Lambda')^{l^{2k_1-2}_{\gamma\delta}}\ldots\sigma_0(\Lambda')^{l^0_{\gamma\delta}}}\, z^\gamma\xi^\delta,$$

where $d_{\gamma\delta}(\Lambda')$ are $\mathbb{C}^{N'}$ valued polynomials in $\mathbb{C}^{K'(2k_1\ell_0)}$ and $l^k_{\gamma\delta}$ nonnegative integers, satisfying the following. For every $H \in \hat{\mathcal{F}}(M,M')$ satisfying (3.4.8), we have

$$\begin{aligned} &\sigma_k((\partial^\alpha H(0))') = \det\big(\partial_{z_{j_p}} F_l(0)\big)_{1\le l,p\le n'} && \text{if } k \text{ is even,}\\ &\sigma_k((\partial^\alpha H(0))') = \overline{\det\big(\partial_{z_{j_p}} F_l(0)\big)}_{1\le l,p\le n'} && \text{if } k \text{ is odd,}\end{aligned} \tag{4.1.15}$$

and

$$H(v^{2k_1}(z,\xi)) \sim \Xi^{\tilde{\jmath}}\left(z,\xi,(\partial^\alpha H(0))'\right), \tag{4.1.16}$$

where $(\partial^\alpha H(0)) = (\partial^\alpha H(0))_{1\le|\alpha|\le 2k_1\ell_0}$ and $(\partial^\alpha H(0)) = ((\partial^\alpha H(0))', (\partial^\alpha H(0))'')$ as explained in Remark 3.4.11. (Recall that $G(z,0)\sim 0$ so that $\partial_z^\beta G(0,0) = 0$ for all $\beta$, i.e. $(\partial^\alpha H(0))'' = 0$.) Indeed, the existence of such functions $\sigma_k$, $1 \le k \le 2k_1 - 1$, and the formal power series $\Xi^{\tilde{\jmath}}$ follows by making repeated use of (3.4.27) for $k = 0,1,\ldots,2k_1-1$, complex conjugating every other equation, and substituting inductively. The rational functions $\sigma_k$ appear naturally after each inductive substitution, and the property (4.1.15) is an immediate consequence of their definition. The form (4.1.14) of $\Xi^{\tilde{\jmath}}$ is a consequence of its construction and (3.4.7). The details are tedious but straightforward, and left to the reader. It should be noted that the rational functions in the right hand side of (4.1.14) depend on $\tilde{\jmath}$, but we have suppressed this dependence to simplify the notation.

In what follows, we shall assume that $k_1 = 2j$ and leave the odd case to the reader. As in the proof of Lemma 4.1.3, we write $\xi = (\xi', z^j, \xi'')$, where $\xi' = (\chi^1, z^1, \ldots, \chi^j)$ and $\xi'' = (\chi^{j+1}, z^{j+1}, \ldots, \chi^{2j})$. In view of Lemma 4.1.3, we can choose $d$ components $y'' = (y''_1,\ldots,y''_d)$ from the components of $\xi''$ such that

$$\operatorname{rk}\left(\frac{\partial u^{2k_1}(z,\xi',z^j,\xi'')}{\partial y''}\circ D_{k_1}\right) = d,$$

where $u^{2k_1}$ is as defined in (4.1.9). After reordering the components of $\xi''$ if necessary, we may write $\xi'' = (x'',y'')$, with $x'' = (x''_1,\ldots,x''_{(k_1-1)n-d})$ and $y''$ as above. We define the linear isomorphism $m : \mathbb{C}^{(k_1-1)n} \to \mathbb{C}^{(k_1-1)n}$ so that

$$D_{k_1}(\xi', z^j) = (0,\xi',z^j,m(\xi')). \tag{4.1.17}$$

Using the decomposition $\xi'' = (x'', y'')$, the mapping $m$ splits in the obvious way as $m = (m_{x''}, m_{y''})$. We shall need the following version of the implicit function theorem with singularities.

**Proposition 4.1.18.** *Let $u(x,t,y)$ be a formal mapping $(\mathbb{C}^{r_1} \times \mathbb{C}^{r_2} \times \mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ such that*

$$u(x,0,0) \sim 0, \quad \operatorname{rk}\left(\frac{\partial u}{\partial y}(x,0,0)\right) = d. \tag{4.1.19}$$

*Then the equation*

$$u(x,t,y) \sim w, \tag{4.1.20}$$

*has a unique solution of the form*

$$y = \Delta(x)\theta\left(x, \frac{t}{\Delta(x)^2}, \frac{w}{\Delta(x)^2}\right), \tag{4.1.21}$$

*where $\theta(t_1, t_2, t_3)$ is a formal mapping $(\mathbb{C}^{r_1} \times \mathbb{C}^{r_2} \times \mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ and*

$$\Delta(x) = \det\left(\frac{\partial u}{\partial y}(x,0,0)\right). \tag{4.1.22}$$

*If, in addition, the mapping $u$ is holomorphic in a neighborhood of the origin, then the mapping $\theta$ is also holomorphic in a neighborhood of the origin and (4.1.21) solves the holomorphic equation $u(x,t,y) = w$ for $(x,t,y,w)$ such that $\Delta(x) \neq 0$ and $|t/\Delta(x)^2| + |w/\Delta(x)^2|$ sufficiently small.*

*Proof.* It follows from the first condition in (4.1.19) that

$$u(x,t,y) = a(x,t,y)t + g(x,t,y)y, \tag{4.1.23}$$

where $a(x,t,y)$ is a $d \times r_2$ matrix of formal power series, $g(x,t,y)$ is a $d \times d$ matrix of formal power series. By expanding $g(x,t,y)$ in $t$ and $y$, we obtain

$$u(x,t,y) = g(x,0,0)y + \left(y^\tau R_j(x,t,y)y\right)_{1 \le j \le d} + \tilde{a}(x,t,y)t, \tag{4.1.24}$$

where each $R_j(x,t,y)$ is a $d \times d$ matrix of formal power series and $\tilde{a}(x,t,y)$ is a $d \times r_2$ matrix of formal power series. Note that

$$g(x,0,0) \sim \frac{\partial u}{\partial y}(x,0,0). \tag{4.1.25}$$

Using Cramer's rule on the equation $u(x,t,y) \sim w$, we obtain, for some $d \times d$ matrix $b(x)$ of formal power series

$$\Delta(x)y + b(x)\left(y^\tau R_j(x,t,y)y\right)_{1 \le j \le d} + b(x)\tilde{a}(x,t,y)t \sim b(x)w, \tag{4.1.26}$$

or, after dividing by $\Delta(x)^2$,

$$(4.1.27)\quad \frac{y}{\Delta(x)} + b(x)\left(\frac{y^{\tau}}{\Delta(x)} R_j(x,t,y)\frac{y}{\Delta(x)}\right)_{1\le j\le d} + b(x)\tilde{a}(x,t,y)\frac{t}{\Delta(x)^2} \sim b(x)\frac{w}{\Delta(x)^2}$$

in $\mathbb{C}[[x,t,y,w,1/\Delta(x)]]$. Put $y' = y/\Delta(x), t' = t/\Delta(x)^2, w' = w/\Delta(x)^2$, and consider the equation

$$(4.1.27)\quad y' + b(x)\big((y')^{\tau} R_j(x,\Delta(x)^2 t', \Delta(x)y')y'\big)_{1\le j\le d} + b(x)\tilde{a}(x,\Delta(x)^2 t', \Delta(x)y')t' \sim b(x)w'.$$

This has a unique formal mapping solution $y' = \theta(x,t',w')$, with

$$\theta : (\mathbb{C}^{r_1}\times\mathbb{C}^{r_2}\times\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0),$$

by the formal implicit function theorem. The conclusion of the proposition in the formal case follows by substituting for $y'$, $t'$, and $w'$.

If $u$ is holomorphic in a neighborhood of 0, then the $\mathbb{C}^d$-valued function $\theta(x,t',w')$ is also holomorphic in a neighborhood of 0, and it is straightforward to verify the last conclusion of Proposition 4.1.18. □

We return again to the proof of Theorem 2.1.5. We may apply Proposition 4.1.18 to the equation

$$(4.1.28)\qquad u^{2k_1}(z,\xi',z^j,m_{x''}(\xi'),y'') \sim w,$$

with $x = (\xi',z^j)$, $t = z$, and $y = y'' - m_{y''}(\xi')$, since the conditions in (4.1.19) are satisfied by Lemma 4.1.3. We conclude that the equation (4.1.28) has a solution of the form

$$(4.1.29)\qquad y'' = \phi\left(\xi',z^j,\frac{z}{\Delta(\xi',z^j)^2},\frac{w}{\Delta(\xi',z^j)^2}\right),$$

where $\phi$ is a formal mapping $(\mathbb{C}^{(k_1+1)n+d},0) \to (\mathbb{C}^d,0)$ and

$$(4.1.30)\qquad \Delta(\xi',z^j) = \det\left(\frac{\partial u^{2k_1}}{\partial y''}\right)(0,\xi',z^j,m(\xi')).$$

Thus, we have

$$(4.1.31)\quad v^{2k_1}\left(z,\xi',z^j,m_{x''}(\xi'),\phi\left(\xi',z^j,\frac{z}{\Delta(\xi',z^j)^2},\frac{w}{\Delta(\xi',z^j)^2}\right)\right) \sim (z,w).$$

Substituting this in (4.1.15), we obtain

(4.1.32) $H(z,w) \sim$

$$\Xi^{\bar{\jmath}}\left(z,\xi',z^{\jmath},m_{x''}(\xi'),\phi\left(\xi',z^{\jmath},\frac{z}{\Delta(\xi',z^{\jmath})^2},\frac{w}{\Delta(\xi',z^{\jmath})^2}\right),(\partial^\alpha H(0))'\right),$$

where $(\partial^\alpha H(0)) = (\partial^\alpha H(0))_{1\le|\alpha|\le 2k_1\ell_0}$ and $(\partial^\alpha H(0)) = ((\partial^\alpha H(0))', (\partial^\alpha H(0))'')$ as explained in Remark 4.1.11. Expanding (4.1.32) as a formal power series in $z$ and $w$, we obtain, using (4.1.14),

$$H(z,w) \sim \sum_{\alpha,\beta} \frac{R_{\alpha\beta}(\xi',z^{\jmath},(\partial^\alpha H(0))')}{\Delta(\xi',z^{\jmath})^{2(|\alpha|+|\beta|)}} z^\alpha w^\beta, \tag{4.1.33}$$

where $R_{\alpha\beta}(\xi',z^{\jmath},\Lambda')$ is of the form

(4.1.34) $R_{\alpha\beta}(\xi',z^{\jmath},\Lambda') \sim$

$$\sum_{\gamma,\delta} \frac{R_{\alpha\beta\gamma\delta}(\Lambda')}{\sigma_{2k_1-1}(\Lambda')^{l^{2k_1-1}_{\alpha\beta\gamma\delta}}\sigma_{2k_1-2}(\Lambda')^{l^{2k_1-2}_{\alpha\beta\gamma\delta}}\dots\sigma_0(\Lambda')^{l^0_{\alpha\beta\gamma\delta}}}(\xi')^\gamma(z^{\jmath})^\delta,$$

for polynomials $R_{\alpha\beta\gamma\delta}(\Lambda')$ on $\mathbb{C}^{K'(2k_1\ell_0)}$ and nonnegative integers $l_{\alpha\beta\gamma\delta}$; we have suppressed the dependence on $\bar{\jmath}$ above to simplify the notation.

We next construct a formal mapping $\hat{\Phi}(Z,\Lambda) = \hat{\Phi}^{\bar{\jmath}}(Z,\Lambda)$, with $Z = (z,w)$ and $\Lambda = (\Lambda',\Lambda'')$, as follows. First set

$$\Gamma(z,w,\xi',z^{\jmath},\Lambda') = \sum_{\alpha,\beta} \frac{R_{\alpha\beta}(\xi',z^{\jmath},\Lambda')}{\Delta(\xi',z^{\jmath})^{2(|\alpha|+|\beta|)}} z^\alpha w^\beta. \tag{4.1.35}$$

Since $\Delta(\xi',z^{\jmath}) \not\sim 0$, we may choose $(\xi'^0, z^{\jmath 0})$ so that the formal power series in $t$,

$$\tilde{\Delta}(t) = \Delta(t\xi'^0, tz^{\jmath 0}) \not\sim 0. \tag{4.1.36}$$

We shall write

(4.1.37)

$$\tilde{\Gamma}(z,w,t,\Lambda') = \Gamma(z,w,t\xi'^0,tz^{\jmath 0},\Lambda') \sim \sum_{\alpha,\beta} \frac{\tilde{R}_{\alpha\beta}(t,\Lambda')}{\tilde{\Delta}(t)^{2(|\alpha|+|\beta|)}} z^\alpha w^\beta,$$

where $\tilde{R}_{\alpha\beta}(t,\Lambda') = R_{\alpha\beta}(t\xi'^0, tz^{\jmath 0},\Lambda')$ are power series in $t$ whose coefficients are rational functions of $\Lambda$ of the form appearing in (4.1.34). Let $r$ be the smallest integer such that $\frac{d^r}{dt^r}\tilde{\Delta}(0) \neq 0$. By division, we may write

$$\tilde{R}_{\alpha\beta}(t,\Lambda') \sim T_{\alpha\beta}(t,\Lambda')\tilde{\Delta}(t)^{2(|\alpha|+|\beta|)} + r_{\alpha\beta}(t,\Lambda'), \tag{4.1.37}$$

where $T_{\alpha\beta}(t, \Lambda')$ is a unique power series in $t$ whose coefficients are finite linear combinations of the coefficients of $\tilde{R}_{\alpha\beta}(t, \Lambda')$, and $r_{\alpha\beta}(t, \Lambda')$ is a polynomial of degree at most $2(|\alpha| + |\beta|)r - 1$ in $t$ whose coefficients coincide with the corresponding coefficients of $\tilde{R}_{\alpha\beta}(t, \Lambda')$. We decompose $\tilde{\Gamma}$ as follows

(4.1.38)

$$\begin{aligned}\tilde{\Gamma}(z, w, t, \Lambda') &\sim \sum_{\alpha,\beta} T_{\alpha\beta}(t, \Lambda') z^\alpha w^\beta + \sum_{\alpha,\beta} \frac{r_{\alpha\beta}(t, \Lambda')}{\tilde{\Delta}(t)^{2(|\alpha|+|\beta|)}} z^\alpha w^\beta \\ &\sim \tilde{\Phi}(z, w, \Lambda') + \sum_{\alpha,\beta} t S_{\alpha\beta}(t, \Lambda') z^\alpha w^\beta + \sum_{\alpha,\beta} \frac{r_{\alpha\beta}(t, \Lambda')}{\tilde{\Delta}(t)^{2(|\alpha|+|\beta|)}} z^\alpha w^\beta,\end{aligned}$$

where, on the second line, we have decomposed $T_{\alpha\beta}(t, \Lambda') \sim T_{\alpha\beta}(0, \Lambda') + tS_{\alpha\beta}(t, \Lambda')$ and where

$$\tilde{\Phi}(z, w, \Lambda') = \sum_{\alpha,\beta} T_{\alpha\beta}(0, \Lambda') z^\alpha w^\beta. \tag{4.1.39}$$

We set $\hat{\Phi}(z, w, \Lambda) := \tilde{\Phi}(z, w, \Lambda')$ (so that $\hat{\Phi}(z, w, \Lambda)$ is independent of $\Lambda''$). We claim that $\hat{\Phi}(z, w, \Lambda)$ is of the form (2.1.6). Indeed, this is easy to check from the above and is left to the reader.

Observe that for each $H \in \hat{\mathcal{F}}(M, M')$ with $\det(\partial F_l/\partial z_{j_p}(0, 0))_{1\le l,p\le n'} \neq 0$ it follows from (4.1.15) that $P^{\bar{J}}(\partial^\alpha H(0)) \neq 0$, where $(\partial^\alpha H(0)) = (\partial^\alpha H(0))_{1\le|\alpha|\le 2k_1\ell_0}$. We claim also that for such an $H$

$$H(z, w) \sim \hat{\Phi}(z, w, (\partial^\alpha H(0))) \sim \tilde{\Phi}(z, w, (\partial^\alpha H(0))'), \tag{4.1.40}$$

with $(\partial^\alpha H(0)) = ((\partial^\alpha H(0))', (\partial^\alpha H(0))'')$ as in Remark 3.4.11. In view of (4.1.33), (4.1.35), (4.1.37), and (4.1.38), we have

$$\begin{aligned}H(z, w) - \hat{\Phi}(z, w, (\partial^\alpha H(0))) &\sim \\ \sum_{\alpha,\beta} &\left( tS_{\alpha\beta}(t, (\partial^\alpha H(0))') + \frac{r_{\alpha\beta}(t, (\partial^\alpha H(0))')}{\tilde{\Delta}(t)^{2(|\alpha|+|\beta|)}} \right) z^\alpha w^\beta.\end{aligned} \tag{4.1.41}$$

Note that each coefficient on the right hand side of (4.1.41) is a Laurent series in $t$ without constant term. Since the left hand side is independent of $t$, we conclude that each coefficient on the right hand side must be zero and, hence, (4.1.40) holds.

Assume now that $M$ and $M'$ are real-analytic. An inspection of the proof above, using the fact that the functions $\Psi_\alpha^{\bar{J}}(Z, \zeta, \zeta', \Lambda)$ of Theorem 3.4.6 are holomorphic near every $(Z, \zeta, \zeta', \Lambda', \Lambda'') = (0, 0, 0, \Lambda_0', 0)$ for each $\Lambda_0'$ satisfying (3.4.10), and the fact that $\hat{\Phi}(Z, \Lambda)$ is independent of $\Lambda''$, shows that $\hat{\Phi}(Z, \Lambda)$ is holomorphic near every $(0, \Lambda_0)$ with $\Lambda_0 = (\Lambda_0', \Lambda_0'')$ such that $\Lambda_0'$ satisfies (3.4.10).

Now, the formal mapping $\hat{\Phi}(Z, \Lambda)$ satisfies all the conclusions of Theorem 2.1.5, with the polynomial $P^{\bar{\jmath}}(\Lambda)$ defined as above, except that $\hat{\Phi}(Z, \Lambda)$, and also the polynomial $P^{\bar{\jmath}}(\Lambda)$, are functions of $\Lambda \in J^{2k_1\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$, and $2k_1$ need not be $\leq d+1$ (although $k_1 \leq d+1$ as noted in Sect. 1.3). We shall address this point, and complete the proof in the next section.

### *4.2. Conclusion of the proof of Theorem 2.1.5*

To complete the proof of the theorem, we shall need the following proposition.

**Proposition 4.2.1.** *Let $F : (\mathbb{C}^l, 0) \to (\mathbb{C}^k, 0)$ be a formal mapping of the form*

$$F(x) = (F_1^0(x) + O(\kappa_1 + 1), \dots, F_k^0(x) + O(\kappa_k + 1)), \tag{4.2.2}$$

*where $F_j^0(x)$ is a homogeneous polynomial of degree $\kappa_j \geq 1$ for $j = 1, \dots, k$. Assume that $\mathrm{Rk}(F^0) = k$, where $F^0 = (F_1^0, \dots, F_k^0)$. Then, for every $\alpha \in \mathbb{Z}_+^k$, there exists a linear form $\mathcal{P}_\alpha : \mathbb{C}^{\sigma(\|\alpha\|)} \to \mathbb{C}$, where*

$$\|\alpha\| = \sum_{j=1}^{k} \kappa_j \alpha_j \tag{4.2.3}$$

*and $\sigma(\nu)$ denotes the number of $\beta \in \mathbb{Z}_+^l$ with $|\beta| \leq \nu$, such that the following holds. For every*

$$g(y) \sim \sum_\alpha g_\alpha y^\alpha \in \mathbb{C}[[y_1, \dots, y_k]], \quad h(x) \sim \sum_\beta h_\beta x^\beta \in \mathbb{C}[[x_1, \dots, x_l]] \tag{4.2.4}$$

*such that*

$$g(F(x)) \sim h(x), \tag{4.2.5}$$

*we have*

$$g_\alpha = \mathcal{P}_\alpha\big((h_\beta)_{|\beta| \leq \|\alpha\|}\big), \quad \forall \alpha \in \mathbb{Z}_+^k. \tag{4.2.6}$$

*Moreover, the coefficients of the linear form $\mathcal{P}_\alpha$ are rational functions of the coefficients of $F$.*

*Proof.* We decompose $g$ into *weighted* homogeneous terms,

$$g(y) \sim \sum_{\nu=0}^{\infty} g^{\nu}(y), \tag{4.2.7}$$

where $g^{\nu}(y)$ is a weighted homogeneous polynomial of degree $\nu$ with respect to the weights $(\kappa_1, \dots, \kappa_k)$, i.e.

$$g^{\nu}(y) \sim \sum_{\|\alpha\|=\nu} g_{\alpha} y^{\alpha}. \tag{4.2.8}$$

We also decompose $h(x)$ into standard homogeneous terms

$$h(x) \sim \sum_{\nu=0}^{\infty} h^{\nu}(x), \tag{4.2.9}$$

where $h_{\nu}(x)$ is a homogeneous polynomial of degree $\nu$. Composing $g$ with $F$ and identifying terms of degree $\nu$ in (4.2.5), we obtain

$$g^{\nu}(F^0(x)) = h^{\nu}(x) + \dots, \tag{4.2.8}$$

where the dots signify terms involving $g_{\alpha}$ for $\|\alpha\| < \nu$. Consider the linear mapping $T_{\nu} : \mathcal{H}_{\nu}^{\kappa}[y] \to \mathcal{H}_{\nu}[x]$, where $\mathcal{H}_{\nu}^{\kappa}[y]$ denotes the space of weighted (with respect to $\kappa = (\kappa_1, \dots, \kappa_k)$) homogeneous polynomials in $y$ of degree $\nu$ and $\mathcal{H}_{\nu}[x]$ the space of homogeneous polynomials in $x$ of degree $\nu$, defined by

$$T_{\nu}(g_{\nu}) = g_{\nu} \circ F^0. \tag{4.2.9}$$

The fact that $\mathrm{Rk}(F^0(x)) = k$ implies that $T_{\nu}$ is injective for each $\nu = 0, 1, \dots$. Hence, it has a left inverse $L_{\nu} : \mathcal{H}_{\nu}[x] \to \mathcal{H}_{\nu}^{\kappa}[y]$. It follows that

$$g^{\nu} = L_{\nu}(h^{\nu} + \dots). \tag{4.2.10}$$

Since, as mentioned above, the dots involve only $g_{\alpha}$ with $\|\alpha\| < \nu$, the proof of Proposition 4.2.1 is easily completed by induction on $\nu$. ▢

We now return again to the proof of Theorem 2.1.5. We shall keep $\tilde{\jmath}$ fixed as in the previous section. Recall, from the proof of Theorem 3.1.9 in §3.3, the notation $v_0^k(z, \chi^1, \dots)$ for the lowest order homogeneous terms (in each component) of the Segre mapping $v^k(z, \chi^1, \dots)$. Since $M$ is of finite type at 0, an inspection of the proof of Theorem 3.1.9 shows that there exists an integer $k_1 \le d + 1$ (also called $k_1$ in the proof of Theorem 3.1.9) such that $\mathrm{Rk}\,(v_0^k) = N$ for $k \ge k_1$. In what follows, we shall assume that $k_1$ is even (and leave the odd case to the reader), and let $\xi = (\chi^1, z^1, \dots) \in \mathbb{C}^{(k_1-1)n}$.

The same argument used to obtain (4.1.16) shows that there exists a $\mathbb{C}^{N'}$ valued formal power series in $(z,\xi)$, $\Xi^{\bar{\jmath}}(z,\xi,\Lambda')$ of the form
(4.2.11)
$$\Xi^{\bar{\jmath}}(z,\xi,\Lambda') \sim \sum_{\gamma,\delta} \frac{d_{\gamma\delta}(\Lambda')}{\sigma_{k_1-1}(\Lambda')^{l_{\gamma\delta}^{k_1-1}}\,\sigma_{k_1-2}(\Lambda')^{l_{\gamma\delta}^{k_1-2}}\ldots\sigma_0(\Lambda')^{l_{\gamma\delta}^{0}}}\, z^\gamma \xi^\delta$$

where $d_{\gamma\delta}(\Lambda')$ are $\mathbb{C}^{N'}$ valued polynomials in $\mathbb{C}^{K'(k_1\ell_0)}$, $l_{\gamma\delta}^{k}$ nonnegative integers, and the rational functions $\sigma_k(\Lambda')$ for $0 \le k \le k_1 - 1$ are the same as those appearing in (4.1.14), satisfying the following.

For every $H \in \hat{\mathcal{F}}(M,M')$ satisfying (3.4.8), we have

$$H(v^{k_1}(z,\xi)) \sim \Xi^{\bar{\jmath}}\left(z,\xi,(\partial^\alpha H(0))'\right), \tag{4.2.12}$$

where $(\partial^\alpha H(0)) = (\partial^\alpha H(0))_{1\le|\alpha|\le k_1\ell_0}$ and $(\partial^\alpha H(0)) = ((\partial^\alpha H(0)', (\partial^\alpha H(0)'')$ as explained in Remark 3.4.11. We shall apply Proposition 4.2.1 with $x = (z,\xi)$, $y = Z = (z,w)$, $F(x) = v^{k_1}(z,\xi)$, $g(y) = H(Z)$, and $h(x) = \Xi^{\bar{\jmath}}(z,\xi,(\partial^\alpha H(0))')$. Since $F^0 = v_0^{k_1}$ has rank $N$ by definition of $k_1$, it follows that the hypotheses on $F$ in Proposition 4.2.1 is satisfied with $\kappa_j = 1$ for $j = 1,\ldots,n$, and $\kappa_{n+j} = \mu_j$ for $j = 1,\ldots,d$, where the $\mu_j$ denote the Hörmander numbers with multiplicity as defined in §3.2. We conclude, using (4.2.11), that for every $\beta \in \mathbb{Z}_+^N$, there is a linear form $\mathcal{P}_\beta$ such that

$$\partial^\beta H(0) = \mathcal{P}_\beta\left(\left(\frac{d_{\gamma\delta}(\Lambda')}{\sigma_{k_1-1}(\Lambda')^{l_{\gamma\delta}^{k_1-1}}\,\sigma_{k_1-2}(\Lambda')^{l_{\gamma\delta}^{k_1-2}}\ldots\sigma_0(\Lambda')^{l_{\gamma\delta}^{0}}}\right)_{|\gamma|+|\delta|\le\|\beta\|}\right), \tag{4.2.13}$$

with $\Lambda' = (\partial^\alpha H(0)')$ as in (4.2.12) (in particular, $|\alpha| \le k_1\ell_0$). Now, let us write $\Lambda' = (\Lambda'_\alpha)$, where $\Lambda'_\alpha$ stands for the $\partial^\alpha H(0)$ part of the jet $j_0^{2k_1\ell_0}(H)$ which appears in $\Lambda'$ (see Remark 3.4.11). By substituting
(4.2.14)
$$\Lambda'_\beta = \mathcal{P}_\beta\left(\left(\frac{d_{\gamma\delta}(\Lambda'_\alpha)_{|\alpha|\le k_1\ell_0}}{\sigma_{k_1-1}(\Lambda')^{l_{\gamma\delta}^{k_1-1}}\,\sigma_{k_1-2}(\Lambda')^{l_{\gamma\delta}^{k_1-2}}\ldots\sigma_0(\Lambda')^{l_{\gamma\delta}^{0}}}\right)_{|\gamma|+|\delta|\le\|\beta\|}\right),$$

for $k_1\ell_0 < |\beta| \le 2k_1\ell_0$ in (4.1.39), we obtain the desired formal mapping $\Phi^{\bar{\jmath}}(Z,\Lambda)$ in Theorem 2.1.5. The polynomial $P^{\bar{\jmath}}$, is obtained, as before, as the product of the numerators of the rational function $\sigma_k$ appearing in (4.1.14), via the substitution given by (4.2.14). The fact that $P^{\bar{\jmath}}(\partial^\alpha H(0)) \ne 0$, for $H \in \hat{\mathcal{F}}(M,M')$ satisfying (3.4.8), follows from (4.1.15) and the observation that the substitution above does not change the values of $\sigma_k(\Lambda)$ when $\Lambda = (\partial^\alpha H(0))$ in view of (4.2.13).

In the case where $M$ and $M'$ are real-analytic, we leave it to the reader to check that substituting (4.2.14) in (4.1.39) (which in this case is holomorphic) yields a holomorphic mapping as described in the theorem. This completes the proof of Theorem 2.1.5. □

### 4.3. *Proofs of Theorems 4, 2.1.9, 2.1.12, and 2.1.14*

*Proof of Theorem* 2.1.9. Let $k_1$, $P^{\tilde{j}}$, $\pi_{\tilde{j}}$, $\Phi^{\tilde{j}}$, for $\tilde{j} = (j_1, \dots, j_{n'})$ with $1 \leq j_1 < \dots < j_{n'} \leq n$, be given by Theorem 2.1.5. Then $\Lambda_0 \in J^{k_1\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$ is in the image of (2.1.10) if and only if $P^{\tilde{j}}(\Lambda_0) \neq 0$, for some $\tilde{j}$, and

$$\Lambda_0 = j_0^{k_1\ell_0}\left(\Phi^{\tilde{j}}(\cdot, \Lambda_0)\right), \tag{4.3.1}$$

$$\rho'\left(\Phi^{\tilde{j}}(Z, \Lambda_0), \overline{\Phi^{\tilde{j}}}(\zeta, \bar{\Lambda}_0)\right) \sim a(Z, \zeta)\rho(Z, \zeta), \tag{4.3.2}$$

for some $d \times d$ matrix $a(Z, \zeta)$ of formal power series, where $\rho$ and $\rho'$ are defining power series for $M$ and $M'$ respectively. In view of (2.1.6), the equation (4.3.1) is a finite set of polynomial equations on $\Lambda_0$. Similarly, by (2.1.6) and elementary linear algebra, (4.3.2) is an infinite set of polynomial equations on $\Lambda_0$ and $\bar{\Lambda}_0$. Thus, the solutions $\Lambda_0$ to (4.3.1) and (4.3.2) form a (possibly empty) real-algebraic subvariety $A_{\tilde{j}}$ of $J^{k_1\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$, and the image of (2.1.10) coincides with $A \setminus B$, where

$$A = \bigcup_{\tilde{j}} A_{\tilde{j}}, \quad B = \bigcap_{\tilde{j}} \{P^{\tilde{j}}(\Lambda) = 0\}. \tag{4.3.3}$$

To see that the image is totally real at each regular point, we pick a regular point $\Lambda_0 \in A_{\tilde{j}} \setminus B$ for some $\tilde{j}$. Thus, there is a unique mapping $H^0 \in \hat{\mathcal{F}}(M, M')$ with $j_0^{k_1\ell_0}(H^0) = \Lambda_0$. By applying the basic identity Theorem 3.4.6, with $Z = 0$ and $\zeta = 0$, to the mapping $H^0$ and using (4.2.13) (cf. [BER3, Lemma 3.7]), we deduce that there exists a rational mapping $T(\Lambda)$, holomorphic near $\Lambda_0$, such that

$$\Lambda = T(\bar{\Lambda}), \tag{4.3.4}$$

holds for each $\Lambda \in A_{\tilde{j}} \setminus B$ near $\Lambda_0$. From this, it is easy to see that $A_{\tilde{j}} \setminus B$ is totally real at $\Lambda_0$. This completes the proof of Theorem 2.1.9. □

*Proof of Theorem* 4. We shall prove Theorem 4 with $(d+1)\ell_0$ replaced by $k_1\ell_0$, where $k_1$ is as above. Clearly, this implies Theorem 4, since $k_1 \leq d+1$. In view of the proof above and Theorem 3, it suffices to show that the mapping (2.1.10) is a homeomorphisms onto $A \setminus B$, with $A$ and $B$ as in the proof of Theorem 2.1.9, when $M$ and $M'$ are real-analytic. This is an easy consequence of Theorem 2.1.5 and the details are left to the reader (see also the proof of [BER3, Theorem 1]). □

*Proofs of Theorems* 2.1.12 *and* 2.1.14. The conclusions of these theorems follow immediately from Theorems 2.1.9 and 4 (or, more precisely, the version of Theorem 4 with $(d+1)\ell_0$ replaced by $k_1\ell_0$ as proved above), since a locally closed subgroup of a Lie group is a Lie subgroup (see e.g [Va]). □

## 5. Smooth perturbations of formal generic submanifolds

### *5.1. Smooth families of submanifolds*

In this section, we shall study the behavior of the series $\Phi^{\tilde{\jmath}}$, given in Theorem 2.1.5, under smooth perturbations of the formal submanifolds $M$ and $M'$. For this, we need the notion of a smooth family of formal generic submanifolds which will be introduced below. A particularly important example, discussed in detail in Sect. 5.2, is the family obtained by taking the formal generic submanifold associated to a given smooth generic submanifold $M$ at varying points $p \in M$.

Let $\rho(Z,\zeta;x) = (\rho_1(Z,\zeta;x),\ldots,\rho_d(Z,\zeta;x))$, where $Z = (Z_1,\ldots,Z_N)$ and $\zeta = (\zeta_1,\ldots,\zeta_N)$, be a smooth family of formal defining series, i.e. each $\rho_j$ is a formal power series in $Z$ and $\zeta$ whose coefficients are smooth functions of $x$ for $x$ in some smooth manifold $X$ and, for each fixed $x \in X$, $\rho(\cdot,\cdot;x)$ is a defining series of a formal generic submanifold, denoted by $M_x$, as explained in Sect. 1. The collection $\{M_x\}$, $x \in X$, will be referred to as a *smooth family* of formal generic submanifolds through 0 in $\mathbb{C}^N$. If $X$ is a real-analytic manifold and the coefficients of $\rho$ depend real-analytically on $x$, then we say that the family is real-analytic. We have the following result.

**Theorem 5.1.1.** *Let $\{M_x\}$, $x \in X$, and $\{M'_y\}$, $y \in Y$, be smooth families of formal generic submanifolds of codimension d and $d'$ through 0 in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively. Assume that $M_{x_0}$, for $x_0 \in X$, is of finite type at 0, that $M'_{y_0}$, for $y_0 \in Y$, is $\ell_0$-nondegenerate for some nonnegative integer $\ell_0$, and that $n \geq n'$, where $n = N-d$ and $n' = N'-d'$. Then there exist open neighborhoods $U \subset X$ and $V \subset Y$ of $x_0$ and $y_0$ respectively, an integer $k_1$ with $1 < k_1 \leq d+1$, such that for each $\tilde{\jmath} = (j_1,\ldots,j_{n'})$ with $1 \leq j_1 < \ldots < j_{n'} \leq n$, there exists a polynomial $P^{\tilde{\jmath}}(\cdot;x,y)$ on $J^{k_1\ell_0}(\mathbb{C}^N,\mathbb{C}^{N'})_{(0,0)}$ whose coefficients depend smoothly on $x,y \in U \times V$*

*and a formal power series in* $Z = (Z_1, \dots, Z_N)$,

$$\Phi^{\bar{\jmath}}(Z, \Lambda; x, y) \sim \sum_{|\alpha|>0} \frac{c_\alpha^{\bar{\jmath}}(\Lambda; x, y)}{P^{\bar{\jmath}}(\Lambda; x, y)^{l_\alpha^{\bar{\jmath}}}} Z^\alpha, \tag{5.1.2}$$

*where* $c_\alpha^{\bar{\jmath}}(\cdot; x, y)$ *are* $\mathbb{C}^{N'}$*-valued polynomials on* $J^{k_1\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$ *whose coefficients depend smoothly on* $x, y \in U \times V$ *and* $l_\alpha^{\bar{\jmath}}$ *are nonnegative integers, with the following property. For every formal CR submersive mapping* $H : (M_x, 0) \to (M'_y, 0)$, *with* $x, y \in U \times V$, *there exists* $\bar{\jmath}$ *as above such that* $P^{\bar{\jmath}}(j_0^{k_1 l_0}(H); x, y) \neq 0$ *if* $k_1$ *is even,* $P^{\bar{\jmath}}(\overline{j_0^{k_1 l_0}(H)}; x, y) \neq 0$ *if* $k_1$ *is odd, and*

$$\begin{aligned} &H(Z) \sim \Phi^{\bar{\jmath}}(Z, j_0^{k_1\ell_0}(H); x, y), \text{ if } k_1 \text{ is even,} \\ &H(Z) \sim \Phi^{\bar{\jmath}}\left(Z, \overline{j_0^{k_1\ell_0}(H)}; x, y\right), \text{ if } k_1 \text{ is odd.} \end{aligned} \tag{5.1.3}$$

*In addition, if* $M_x$ *and* $M'_y$, *for some* $x, y \in U \times V$, *are real-analytic, then for every* $\bar{\jmath}$ *as above and* $\Lambda_0$ *with* $P(\pi_{\bar{\jmath}}(j_0^{k_1\ell_0,1}(\Lambda_0); x, y); x, y) \neq 0$ *the series* (5.1.3) *converges uniformly for* $(Z, \Lambda)$ *near* $(0, \Lambda_0)$ *in* $\mathbb{C}^N \times J^{k_1\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$. *If the families* $\{M_x\}$, $x \in U$, *and* $\{M'_y\}$, $y \in Y$, *are real-analytic, then the dependence of* $P(\cdot; x, y)$, $\pi_{\bar{\jmath}}(\cdot; x, y)$, *and* $c_\alpha^{\bar{\jmath}}(\cdot, \cdot; x, y)$ *above on* $x, y \in U \times V$ *is real-analytic for* $x, y \in U \times V$.

*Proof.* For the proof of Theorem 5.1.1, we shall need the following version of the formal implicit function theorem in which the dependence of the coefficients in the solution on the coefficients in the equation is described. The proof, which consists of applying the usual (formal) implicit function theorem and identifying coefficients in the equation, is left to the reader.

**Lemma 5.1.4.** *Let* $k$ *and* $m$ *be nonnegative integers. Then there exist polynomials* $\mathcal{P}_\gamma$, *for* $\gamma \in \mathbb{Z}_+^k$, *with the following property. For every formal power series mapping* $F : (\mathbb{C}^k \times \mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$,

$$F(x, y) \sim \sum_{\alpha,\beta} a_{\alpha\beta} x^\alpha y^\beta, \tag{5.1.5}$$

*with* $x = (x_1, \dots, x_k)$ *and* $y = (y_1, \dots, y_m)$, *such that* $\partial F/\partial y(0, 0)$ *is invertible, there is a unique power series solution* $y = f(x)$ *of the equation* $F(x, y) \sim 0$, *where* $f : (\mathbb{C}^k, 0) \to (\mathbb{C}^m, 0)$ *is of the form*

$$f(x) \sim \sum_\gamma b_\gamma x^\gamma \tag{5.1.6}$$

*with*

$$(5.1.7) \qquad b_\gamma = \frac{\mathcal{P}_\gamma\left((a_{\alpha\beta})_{|\alpha|+|\beta|\le|\gamma|}\right)}{\det((a_k^j)_{1\le j,k\le m})^{2|\gamma|-1}}, \quad \forall \gamma \in \mathbb{Z}_+^k,$$

*where* $a_k^j = \partial F_j/\partial y_k(0,0)$.

We return to the proof of Theorem 5.1.1. A consequence of Lemma 5.1.4 (and the construction of normal coordinates; see [CM], [BJT], or [BER4, Chapter IV.2]) is that given a smooth family $\{M_x\}, x \in X$, of formal generic submanifolds through 0 in $\mathbb{C}^N$ and $x_0 \in X$, there is a formal change to normal coordinates $(z(Z;x), w(Z;x))$ whose coefficients depend smoothly on $x \in X$ near $x_0$ such that $M_x$, in these coordinates, is defined by $w - Q(z,\chi,\tau;x)$, where $Q(z,\chi,\tau;x)$ is a $d$-vector of formal power series in $(z,\chi,\tau)$ satisfying (1.12) and whose coefficients depend smoothly on $x \in X$ near $x_0$. Now, Theorem 5.1.1 follows by a detailed inspection of the proof of Theorem 2.1.5 and repeated use of Lemma 5.1.4. The details are omitted. □

It follows from Theorem 5.1.1 and the proof of Theorem 2.1.9 that, under the assumptions of Theorem 5.1.1, the defining equations of the images

$$j^{k_1\ell_0}(\hat{\mathcal{F}}(M_x, M'_y)) \subset J^{k_1\ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})$$

depend smoothly on $(x,y) \in X \times Y$ near $(x_0, y_0)$. Hence, by applying [BER3, Lemma 5.1] (which essentially is the "no small subgroups" property of Lie groups), we obtain in particular the following corollary of Theorem 5.1.1.

**Theorem 5.1.8.** *Let $\{M_x\}$, $x \in X$, be a smooth family of formal generic submanifolds of codimension d through 0 in $\mathbb{C}^N$. Assume that $M_{x_0}$, for $x_0 \in X$, is of finite type and $\ell_0$-nondegenerate, for some nonnegative integer $\ell_0$, at 0. Let $k_1$ be the integer obtained by applying Theorem 2.1.12 to $M_{x_0}$, and assume that the Lie group $j^{k_1\ell_0}(\hat{\mathcal{F}}(M_{x_0}, M_{x_0})) \subset G^{k_1\ell_0}(\mathbb{C}^N)_0$ is discrete. Then there is an open neighborhood $U \subset X$ of $x_0$ such that the Lie groups $j^{k_1\ell_0}(\hat{\mathcal{F}}(M_x, M_x)) \subset G^{k_1\ell_0}(\mathbb{C}^N)_0$, for $x \in U$, are also discrete.*

Theorem 5.1.8 implies e.g. that if $\hat{\mathcal{F}}(M_{x_0}, M_{x_0})$ consists of only the identity, then for $x \in X$ near $x_0$ there are no formal automorphisms "near the identity" (in the sense of jets of order $k_1\ell_0$) in $\hat{\mathcal{F}}(M_x, M_x)$. If the family $\{M_x\}$ consists of real-analytic submanifolds, then Theorem 5.1.9 (combined with Theorems 3 and 2.1.14) yields the following result. If $\mathrm{Aut}(M_{x_0}, 0)$ is discrete (in the natural topology), then $\mathrm{Aut}(M_x, 0)$ is discrete for all $x \in X$ near $x_0$.

### *5.2. Dependence on the base point in a smooth generic submanifold*

We now turn to the particular example of a smooth family obtained by taking the formal generic submanifolds associated to points $p$ on a given smooth generic submanifold $M \subset \mathbb{C}^N$. More precisely, let $M \subset \mathbb{C}^N$ be a smooth generic submanifold with defining function $\rho(Z, \bar{Z})$ near a distinguished point $p_0 \in M$. We shall consider the smooth family $\{M_p\}$, for $p$ in a neighborhood $U$ of $p_0$ in $M$, of formal generic submanifolds through 0 in $\mathbb{C}^N$ defined by the Taylor series at 0 of the smooth defining functions

$$\rho(Z, \zeta; p) := \rho(Z + p, \zeta + \bar{p}), \quad p \in U. \tag{5.2.1}$$

For smooth generic submanifolds $M \subset \mathbb{C}^N$, $M' \subset \mathbb{C}^{N'}$ and points $p \in M$, $p' \in M'$, we denote by $\hat{\mathcal{F}}(M, p; M', p')$ the set of formal mappings $H : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ such that $Z \mapsto H(Z + p) - p'$ maps $M_p$ into $M'_{p'}$ and is CR submersive. Here, a formal mapping $H : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ is such that the components of $H = (H_1, \dots, H_{N'})$ are formal power series in $Z - p$ with constant term $H(p)$ equal to $p'$. We shall need some more notation.

We denote by $E(\mathbb{C}^N, \mathbb{C}^{N'})_{(Z,Z')}$ the set of germs at $Z$ of holomorphic mappings $(\mathbb{C}^N, Z) \to (\mathbb{C}^{N'}, Z')$ and by $\hat{E}(\mathbb{C}^N, \mathbb{C}^{N'})_{(Z,Z')}$ the set of formal mappings $(\mathbb{C}^N, Z) \to (\mathbb{C}^{N'}, Z')$. We denote by $E(\mathbb{C}^N, \mathbb{C}^{N'})$ the disjoint union

$$E(\mathbb{C}^N, \mathbb{C}^{N'}) = \bigcup_{Z \in \mathbb{C}^N, Z' \in \mathbb{C}^{N'}} E(\mathbb{C}^N, \mathbb{C}^{N'})_{(Z,Z')}, \tag{5.2.2}$$

and use similar notation for $\hat{E}(\mathbb{C}^N, \mathbb{C}^{N'})$. We define $\hat{\mathcal{F}}(M, M')_{(U,U')} \subset \hat{E}(\mathbb{C}^N, \mathbb{C}^{N'})$, for open subsets $U \subset M$ and $U' \subset M'$, to be

$$\hat{\mathcal{F}}(M, M')_{(U,U')} = \bigcup_{p \in U, p' \in U'} \hat{\mathcal{F}}(M, p; M', p'), \tag{5.2.3}$$

and $\mathcal{F}(M, M')_{(U,U')}$ similarly. We equip $E(\mathbb{C}^N, \mathbb{C}^{N'})_{(Z,Z')}$ with the natural inductive limit topology, and $E(\mathbb{C}^N, \mathbb{C}^{N'})$ with the topology it inherits by the trivialization

$$E(\mathbb{C}^N, \mathbb{C}^{N'}) \cong \mathbb{C}^N \times \mathbb{C}^{N'} \times E(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)} \tag{5.2.4}$$

defined by taking a germ $H$ at $Z_0$ with $H(Z_0) = Z'_0$ to $(Z_0, Z'_0, H_0) \in \mathbb{C}^N \times \mathbb{C}^{N'} \times E(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$, where

$$H_0(Z) = H(Z + Z_0) - Z'_0. \tag{5.2.5}$$

For a positive integer $k$, we shall denote by $J^k(\mathbb{C}^N, \mathbb{C}^{N'})$ the complex manifold of $k$-jets of holomorphic mappings $\mathbb{C}^N \to \mathbb{C}^{N'}$, i.e.

$$J^k(\mathbb{C}^N, \mathbb{C}^{N'}) = \bigcup_{Z \in \mathbb{C}^N, Z' \in \mathbb{C}^{N'}} J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(Z,Z')} \tag{5.2.6}$$

where $J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(Z,Z')}$ denotes the space of $k$-jets at $Z$ of holomorphic mappings taking $Z$ to $Z'$. We have a similar trivialization

$$J^k(\mathbb{C}^N, \mathbb{C}^{N'}) \cong \mathbb{C}^N \times \mathbb{C}^{N'} \times J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}, \tag{5.2.7}$$

and we have the jet mapping $j^k : \hat{E}(\mathbb{C}^N, \mathbb{C}^{N'}) \to J^k(\mathbb{C}^N, \mathbb{C}^{N'})$. (See e.g. [GG] or [BER4, Chapter XII]; cf. also Sect. 2 above.) We shall use the trivialization (5.2.7) and refer to $(Z, Z', \Lambda) \in \mathbb{C}^N \times \mathbb{C}^{N'} \times J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(0,0)}$ as coordinates for $J^k(\mathbb{C}^N, \mathbb{C}^{N'})$.

We shall also use the notation $J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(U,U')} \subset J^k(\mathbb{C}^N, \mathbb{C}^{N'})$ for the submanifold defined by

$$J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(U,U')} = \bigcup_{p \in U, p' \in U'} J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(p,p')}, \tag{5.2.8}$$

where $U$ and $U'$ are open subsets of $M$ and $M'$ respectively. Observe that $j^k$ maps $\hat{\mathcal{F}}(M, M')_{(U,U')}$ into $J^k(\mathbb{C}^N, \mathbb{C}^{N'})_{(U,U')}$.

We have the following corollary of Theorem 5.1.1, whose proof is similar to those of Theorems 4 and 2.1.9, and is left to the reader.

**Theorem 5.2.9.** *Let $M$ and $M'$ be smooth generic submanifolds through $p_0 \in \mathbb{C}^N$ and $p'_0 \in \mathbb{C}^{N'}$ respectively, such that $M$ is of finite type at $p_0$ and $M'$ is $\ell_0$-nondegenerate at $p'_0$ for some integer $\ell_0$. Then there are open neighborhoods $U \subset M$ and $U' \subset M'$ of $p_0$ and $p'_0$ respectively, an integer $k_1$ depending only on $M$ with $1 < k_1 \le d+1$ where $d$ denotes the codimension of $M$, a finite collection $b_1(p, p', \Lambda), \dots, b_l(p, p', \Lambda)$ of polynomials in $\Lambda$ and a countable collection $\{a_j(p, p', \Lambda, \bar{\Lambda})\}_{j=1}^{\infty}$ of polynomials in $(\Lambda, \bar{\Lambda})$ whose coefficients are smooth functions of $(p, p') \in U \times U'$ with the following property. The image of the mapping*

$$j^{k_1 \ell_0} : \hat{\mathcal{F}}(M, M')_{(U,U')} \to J^{k_1 \ell_0}(\mathbb{C}^N, \mathbb{C}^{N'})_{(U,U')} \tag{5.2.10}$$

*coincides with the locally closed set*

$$\{a_j(p, p', \Lambda, \bar{\Lambda}) = 0,\ j = 1, 2, \dots\} \setminus \{b_k(p, p', \Lambda) = 0,\ k = 1, 2, \dots l\}. \tag{5.2.11}$$

*If, in addition, $M$ and $M'$ are real-analytic, then the coefficients of $a_j(p, p', \Lambda, \bar{\Lambda})$ and $b_k(p, p', \Lambda)$ depend real-analytically on $(p, p') \in U \times U'$, the sets*

$$\begin{aligned} A &= \{a_j(p, p', \Lambda, \bar{\Lambda}) = 0,\ j = 1, 2, \dots\}, \\ B &= \{b_k(p, p', \Lambda) = 0,\ k = 1, 2, \dots l\} \end{aligned} \tag{5.2.12}$$

*are real-analytic subvarieties of* $J^{k_1\ell_0}(\mathbb{C}^N,\mathbb{C}^{N'})_{(U,U')}$, *and the mapping* (5.2.10) *(where* $\hat{\mathcal{F}}(M,M')_{(U,U')} = \mathcal{F}(M,M')_{(U,U')}$ *by Theorem* 3*) is a homeomorphism onto its image* $A \setminus B$.

## 6. Remarks on the algebraic mapping problem

We shall conclude this paper by showing how the arguments presented here can be applied to the algebraic mapping problem; see [BR], [BER1], [M], [Z3], [CMS] for recent work. This problem consists, loosely speaking, of finding conditions which imply that any holomorphic mapping $H : \mathbb{C}^N \to \mathbb{C}^{N'}$, defined near some point $p_0 \in \mathbb{C}^N$ and mapping a given real-algebraic submanifold $M \subset \mathbb{C}^N$ with $p_0 \in M$ into another $M' \subset \mathbb{C}^{N'}$, is algebraic. Recall that a holomorphic mapping is called algebraic if all its components are algebraic functions, and a real submanifold is called real algebraic if it is contained in a real algebraic subvariety of the same dimension (see also [BER4, Chapter V]).

Since this problem is not the main topic of the present paper, we shall refer the reader to the papers mentioned above for a more detailed discussion and history of the problem. We give here only the following new result.

**Theorem 6.1.** *Let* $M \subset \mathbb{C}^N$ *be a real algebraic, generic, and connected submanifold which is of finite type at some point. Let* $M' \subset \mathbb{C}^{N'}$ *be a real algebraic, generic submanifold which is holomorphically nondegenerate and of finite type at every point. Then any holomorphic mapping* $H : \mathbb{C}^N \to \mathbb{C}^{N'}$, *which is defined near some point* $p_1 \in M$, *maps* $M$ *into* $M'$, *and which is CR submersive at some point* $p_2 \in M$, *is algebraic.*

*Proof.* Let $H$ be a holomorphic mapping as in the theorem above. We denote by $\Omega \subset \mathbb{C}^N$ a neighborhood of the point $p_1$ in which $H$ is holomorphic. It follows from the assumptions in the theorem that $M \cap \Omega$ is of finite type outside a proper real-analytic subvariety. Since the restriction of $H$ to $M \cap \Omega$ is CR submersive outside a proper real-analytic subvariety, it follows that there is a point $p_3 \in M \cap \Omega$ at which $M$ is of finite type and $H$ is CR submersive. Moreover, since $M'$ is of finite type at $H(p_3) \in M'$, it follows from Corollary 1.27 that $H$ maps a neighborhood $U \subset M$ of $p_3$ onto a neighborhood $U' \subset M'$ of $H(p_3)$. We deduce, using [BER1, Proposition 1.3.1] (see also [BER4, Theorem 11.5.1]) and the fact that $M'$ is holomorphically nondegenerate, that there is a point $p_4 \in U$ at which $M$ is of finite type, $H$ is CR submersive, and for which $M'$ is finitely nondegenerate at $H(p_4)$. Thus, Theorem 2.1.5 applies with $p_0 = p_4$. Now, an inspection of the proof of Theorem 2.1.5 shows that the functions $\Phi^{\tilde{J}}$ are algebraic when $M$ and $M'$ are real algebraic, and hence it follows from (2.1.8) that $H$ is algebraic. $\square$

*Remark* 6.2. The crucial result used in the proof above, after applying Corollary 1.27, is the following consequence of Theorem 2.1.5 (in the algebraic setting): *If $M \subset \mathbb{C}^N$ is a real algebraic, generic submanifold which is of finite type at $p_0 \in M$ and $M' \subset \mathbb{C}^{N'}$ is a real algebraic, generic submanifold which is finitely nondegenerate at $p_0' \in M'$, then any holomorphic mapping $H : (\mathbb{C}^N, p_0) \to (\mathbb{C}^{N'}, p_0')$ which maps $M$ into $M'$ and is CR submersive at $p_0$ is algebraic.* We should point out that this result also follows from [Z3, Theorem 1.6]. However, the approach in [Z3] differs from the one in this paper.

## References


[BER1] M. S. Baouendi, P. Ebenfelt, L. P. Rothschild, Algebraicity of holomorphic mappings between real algebraic sets in $\mathbb{C}^n$, Acta Math. **177** (1996), 225–273.

[BER2] M. S. Baouendi, P. Ebenfelt, L. P. Rothschild, CR automorphisms of real analytic CR manifolds in complex space, Comm. Anal. Geom. **6** (1998), 291–315.

[BER3] M. S. Baouendi, P. Ebenfelt, L. P. Rothschild, Parametrization of local biholomorphisms of real analytic hypersurfaces, Asian J. Math. **1** (1997), 1–16.

[BER4] M. S. Baouendi, P. Ebenfelt, L. P. Rothschild, Real Submanifolds in Complex Space and their Mappings, Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1999.

[BJT] M. S. Baouendi, H. Jacobowitz, F. Treves, On the analyticity of CR mappings, Ann. Math. **122** (1985), 365–400.

[BR] M. S. Baouendi, L. P. Rothschild, Mappings of real algebraic hypersurfaces, J. Amer. Math. Soc. **8** (1995), 997–1015.

[B] V. K. Beloshapka, A uniqueness theorem for automorphisms of a nondegenerate surface in complex space, Math. Notes **47** (1990), 239–242.

[BS] D. Burns, Jr., S. Shnider, Real hypersurfaces in complex manifolds, Proc. Symp. Pure Math., Vol XXX, part 2, Amer. Math. Soc., Providence, R.I., 1977.

[EC1] E. Cartan, Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes, I, Ann. Math. Pura Appl. **11** (1932), 17–90 (or Oeuvres II, 1231–1304).

[EC2] E. Cartan, Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes, II, Ann. Scoula Norm. Sup. Pisa **1** (1932), 333–354 (or Oeuvres III, 1217–1238)

[HC] H. Cartan, Sur les groupes de transformations analytiques, Act. Sci. et Int., Hermann, Paris, 1935.

[CM] S.S. Chern, J. K. Moser, Real hypersurfaces in complex manifolds, Acta Math. **133** (1974), 219–271.

[CNSW] M. Christ, A. Nagel, E. M. Stein, S. Wainger, Singular and maximal Radon transforms; Analysis and geometry (1998), (preprint).

[CMS] B. Coupet, F. Meylan, A. Sukhov, Holomorphic maps of algebraic CR manifolds, Internat. Math. Res. Notices (1999), no. 1, 1–29.

[DF] K. Diederich, J.-E. Fornaess, Proper holomorphic mappings between real-analytic pseudoconvex domains in $\mathbb{C}^n$, Math. Ann. **282** (1988), 681–700.

[DW] K. Diederich, S. Webster, A reflection principle for degenerate hypersurfaces, Duke Math. J. **47** (1980), 835–843.

[DP] K. Diederich, S. Pinchuk, Proper holomorphic maps in dimension 2 extend, Indiana Univ. Math. J. **44** (1995), 1089–1126.

[F] F. Forstnerič, Extending proper holomorphic mappings of positive codimension, Invent. Math. **95** (1989), 31–62.

[GG] M. Golubitsky, V. Guillemin, Stable Mappings and Their Singularities, Springer-Verlag, Berlin, 1973.

[Hu] X. Huang, Schwarz reflection principle in complex spaces of dimension two, Comm. Partial Differential Equations **21** (1996), 1781–1828.

[Ka] S. Kaneyuki, On the automorphism group of homogeneous bounded domains, J. Fac. Sci. Univ. Tokyo **14** (1967), 89–130.

[Kr] N. G. Kruzhilin, Description of the local automorphism groups of real hypersurfaces, Proceedings of the International Congress of Mathematicians, Berkeley 1986, Amer. Math. Soc., Providence, R. I., 1987, 749–758.

[M] N. Mir, Germs of holomorphic mappings between real algebraic hypersurfaces, Ann. Inst. Fourier (Grenoble) **48** (1998), 1025–1043.

[Se] B. Segre, Intorno al problem di Poincaré della rappresentazione pseudo-conform, Rend. Acc. Lincei **13** (1931), 676–683.

[St1] N. Stanton, Infinitesimal CR automorphisms of rigid hypersurfaces in $\mathbb{C}^2$, J. Geom. Anal. **1** (1991), 231–267.

[St2] N. Stanton, Infinitesimal CR automorphisms of real hypersurfaces, Amer. J. Math. **118** (1996), 209–233.

[Ta] N. Tanaka, On the pseudo-conformal geometry of hypersurfaces of the space of $n$ complex variables, J. Math. Soc. Japan **14** (1962), 397–429.

[Tu] A. Tumanov, Finite-dimensionality of the group of CR automorphisms of a standard CR manifold, and proper holomorphic mappings of Siegel domains, Izvestia Akad. Nauk SSSR, Ser. Mat. **52** (1988); Math. USSR Izvestia **32** (1989), 655–662.

[TH] A. Tumanov, G. M. Henkin, Local characterization of holomorphic automorphisms of Siegel domains, Funktsional. Anal. i Prilozhen **17** (1983), 49–61 English transl. in Functional Anal. Appl. **17** (1983).

[Va] V. S. Varadarajan, Lie Groups, Lie Algebras, and their Representations, Prentice Hall, Inc, Englewood Cliffs, NJ, 1974.

[Vi] A. G. Vitushkin, Holomorphic mappings and the geometry of hypersurfaces, Encyclopaedia of Mathematical Sciences, Vol. 7, Several Complex Variables I, Springer-Verlag, Berlin, 1985, pp. 159–214.

[W1] S. M. Webster, On the mapping problem for algebraic real hypersurfaces, Invent. Math. **43** (1977), 53–68.

[W2] S. M. Webster, On the transformation group of a real hypersurface, Trans. Amer. Math. Soc. **231** (1977), 179–190.

[W3 ] S. M. Webster, On the reflection principle in several complex variables, Proc. Amer. Math. Soc. **71** (1978), 26–28.

[Z1] D. Zaitsev, On the automorphism groups of algebraic bounded domains, Math. Ann. **302** (1995), 105–129.

[Z2] D. Zaitsev, Germs of local automorphisms of real-analytic CR structures and analytic dependence on $k$-jets, Math. Research Lett. **4** (1997), 823–842.

[Z3] D. Zaitsev, Algebraicity of local holomorphisms between real-algebraic submanifolds of complex spaces (1998), (preprint).