A REMARK ON HYPOEILLIPTICITY OF HOMOGENEOUS IN Variant DIFFERENTIAL OPERATORS ON NILPOTENT LIE GROUPS

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In this note we use a recent result of Geller [2] to reformulate the criterion of Helffer-Nourrigat [3] for hypoellipticity of homogeneous left invariant differential operators on graded nilpotent Lie groups.

Let \( g \) be a graded nilpotent Lie algebra, i.e. \( g = g_1 + g_2 + \cdots + g_r \), a vector space direct sum, with \( [g_i, g_j] \subseteq g_{i+j} \). Let \( G \) be its corresponding simply connected Lie group and \( U(g) \) its universal enveloping algebra, identified with the space of all left invariant differential operators on \( G \). Finally, let \( \hat{G} \) be the set of all irreducible unitary representations of \( G \).

The family of dilations \( \delta_s, s > 0 \), on \( g \) defined by \( \delta_s |g_1 = s^1 \) extends, via the exponential map, to dilations \( \delta_s \) which are automorphisms of \( G \). An operator \( L \in U(g) \) is
homogeneous of degree $m$ if $L(f * \delta_s) = s^m(Lf) * \delta_s$, $s > 0$.

$L$ is hypoelliptic if for any open $U \subset G$ and any distributions $u, f$ on $U$, $Lu = f$ and $f \in C^\infty(U)$ imply $u \in C^\infty(U)$ also.

Helffer and Nourrigat [3] proved the following elegant characterization of hypoelliptic operators. (See also [1]).

**Theorem A.** (Helffer-Nourrigat [3]). Let $g$ be a graded nilpotent Lie algebra and $L \in U(g)$ homogeneous of degree $m$.

Then $L$ is hypoelliptic if and only if $\pi(L)$ is injective for all nontrivial $\pi \in \hat{G}$.

Recently Geller [2] proved the following (generalizing a special case of Koranyi and Stanton [4]).

**Theorem B.** (Geller [2]). Let $g$ be a graded nilpotent Lie algebra and $L \in U(g)$ homogeneous and hypoelliptic. Then any function $f$ on $G$ which is of polynomial growth and annihilated by $L$ is a polynomial.

From Theorems A and B we may obtain the following criterion for hypoellipticity.

**Theorem C.** Let $g$ be a graded nilpotent Lie algebra and $L \in U(g)$ homogeneous. Then $L$ is hypoelliptic if and only if there is no nonconstant bounded function $h$ on $G$ such that $Lh = 0$.

**Proof.** One implication is a special case of Geller's theorem cited above. Hence it suffices to prove that if there
is no nonconstant $h$ with $Lh = 0$, then $L$ is hypoelliptic. By the Helffer-Nourrigat criterion (Theorem A) it suffices to show that there is no nontrivial $\pi \in \hat{G}$ for which $\pi(L)$ is not injective. Suppose such a $\pi$ were to exist with

$$\pi(L)v = 0, \ 0 \neq v \in H^\pi,$$

where $H^\pi$ is the Hilbert space on which $\pi$ is realized. Then, as in [6, Lemma 4.6] and [5, Theorem 8.1] we may construct $h$ as follows. For any $s > 0$ let $\pi_s \in \hat{G}$ be defined by

$$\pi_s(g) = \pi(5_sg), \ g \in G.$$  By homogeneity of $L$, $\pi_s(L)v = 0$ for all $s$. Then if $h$ is defined by

$$h(g) = \int_{-\infty}^{\infty} \pi_s(g)v, v > s^N ds,$$

for $N$ sufficiently large, one may easily check that $h$ is bounded, non constant, and $Lh = 0$. This proves Theorem C.

It would be interesting to find a proof of Theorem C not using representation theory.

REFERENCES


de Lie nilpotent gradé", Comm. P.D.E. 4, no. 8 (1979)
899-958.

complex hypoelliptic operators" (preprint).

operators constructed from vector fields", Comm. P.D.E. 4
(6), (1979) 645-699.

operators and nilpotent groups" Acta Math 137 (1976)
247-320.

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