COMM. IN PARTIAL DIFFERENTIAL EQUATIONS, 8(15), 1679-1682 (1983)

# A REMARK ON HYPOELLIPTICITY OF HOMOGENEOUS INVARIANT DIFFERENTIAL OPERATORS ON NILPOTENT LIE GROUPS

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In this note we use a recent result of Geller [2] to reformulate the criterion of Helffer-Nourrigat [3] for hypoellipticity of homogeneous left invariant differential operators on graded nilpotent Lie groups.

Let $\mathfrak{g}$ be a graded nilpotent Lie algebra, i.e. $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 + \dots + \mathfrak{g}_r$, a vector space direct sum, with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. Let $G$ be its corresponding simply connected Lie group and $U(\mathfrak{g})$ its universal enveloping algebra, identified with the space of all left invariant differential operators on $G$. Finally, let $\hat{G}$ be the set of all irreducible unitary representations of $G$.

The family of dilations $\delta_s$, $s > 0$, on $\mathfrak{g}$ defined by $\delta_s|\mathfrak{g}_i = s^i$ extends, via the exponential map, to dilations $\delta_s$ which are automorphisms of $G$. An operator $L \in U(\mathfrak{g})$ is



 0360-5302/83/0815-1679$3.50/0

<u>homogeneous</u> <u>of</u> <u>degree</u> $m$ if $L(f \circ \delta_s) = s^m(Lf) \circ \delta_s$, $s > 0$. $L$ is <u>hypoelliptic</u> if for any open $U \subset G$ and any distributions $u,f$ on $U$, $Lu = f$ and $f \in C^\infty(U)$ imply $u \in C^\infty(U)$ also. Helffer and Nourrigat [3] proved the following elegant characterization of hypoelliptic operators. (See also [1]).

<u>Theorem</u> A. (Helffer-Nourrigat [3]). <u>Let</u> $\mathfrak{g}$ <u>be a graded nilpotent Lie algebra and</u> $L \in U(\mathfrak{g})$ <u>homogeneous of degree</u> $m$. <u>Then</u> $L$ <u>is hypoelliptic if and only if</u> $\pi(L)$ <u>is injective for all nontrivial</u> $\pi \in \hat{G}$.

Recently Geller [2] proved the following (generalizing a special case of Koranyi and Stanton [4]).

<u>Theorem</u> B. (Geller [2]). <u>Let</u> $\mathfrak{g}$ <u>be a graded nilpotent Lie algebra and</u> $L \in U(\mathfrak{g})$ <u>homogeneous and hypoelliptic</u>. <u>Then any function</u> $f$ <u>on</u> $G$ <u>which is of polynomial growth and annihilated by</u> $L$ <u>is a polynomial</u>.

From Theorems A and B we may obtain the following criterion for hypoellipticity.

<u>Theorem</u> C. <u>Let</u> $\mathfrak{g}$ <u>be a graded nilpotent Lie algebra and</u> $L \in U(\mathfrak{g})$ <u>homogeneous</u>. <u>Then</u> $L$ <u>is hypoelliptic if and only if there is no nonconstant bounded function</u> $h$ <u>on</u> $G$ <u>such that</u> $Lh = 0$.

<u>Proof</u>. One implication is a special case of Geller's theorem cited above. Hence it suffices to prove that if there

is no nonconstant $h$ with $Lh = 0$, then $L$ is hypoelliptic. By the Helffer-Nourrigat criterion (Theorem A) it suffices to show that there is no nontrivial $\pi \in \hat{G}$ for which $\pi(L)$ is not injective. Suppose such a $\pi$ were to exist with

$$\pi(L)v = 0, \quad 0 \neq v \in \mathcal{H}_\pi ,$$

where $\mathcal{H}_\pi$ is the Hilbert space on which $\pi$ is realized. Then, as in [6, Lemma 4.6] and [5, Theorem 8.1] we may construct $h$ as follows. For any $s > 0$ let $\pi_s \in \hat{G}$ be defined by $\pi_s(g) = \pi(\delta_s g)$, $g \in G$. By homogeneity of $L$, $\pi_s(L)v = 0$ for all $s$. Then if $h$ is defined by

$$h(g) = \int_1^\infty < \pi_s(g)v,v> s^{-N} ds ,$$

for $N$ sufficiently large, one may easily check that $h$ is bounded, non constant, and $Lh = 0$. This proves Theorem C.

It would be interesting to find a proof of Theorem C not using representation theory.

## REFERENCES

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Received July 1983