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Remarks on the Generic Rank of a CR Mapping

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ABSTRACT. We study germs of smooth CR mappings between embedded real hypersurfaces in complex spaces of the same dimension. In particular, we are interested in the generic rank of such mappings. If $H : M \to M'$ is a CR map between two hypersurfaces M and M', we prove that if M' does not contain any germ of a holomorphic manifold then either H is constant or the generic rank of H is odd. We also prove that if there is no formal holomorphic vector field tangent to M, then either H is constant or generically H is a local diffeomorphism. It follows, as a special case, that if M and M' are of D-finite type (in the sense of D'Angelo) then H is either constant or is generically a local diffeomorphism.

Introduction and statement of results

Let M and M' be two smooth hypersurfaces in $\mathbf{C}^{n+1}$ and H a germ of a CR mapping from M into M' at a point $p_0 \in M$. If M is minimal at p_0, i.e., M contains no germ through p_0 of a complex holomorphic hypersurface, then by a result of Trépreau [8] the mapping H extends holomorphically to one side of M near p_0. If we regard H as a mapping from the real manifold M into the real manifold M', we denote by $\mathrm{rk}_M H'(p)$ the rank of this map at $p \in M$. It follows (see Section 1) that H is of maximal rank in a dense open subset of a neighborhood of p_0 in M. We call this rank the *generic rank* of H. In particular, the generic rank of H is $2n+1$ if and only if the Jacobian determinant of H at p, written $Jac(H)(p)$, does not vanish identically in a neighborhood of p_0. In this paper we give conditions on M and M' that force the generic rank to be odd, even, or equal to $2n+1$. We prove, in particular, (Corollary 0.4) that if M and M' are of D-finite type, in the sense of D'Angelo [1], then H is either constant or its generic rank is $2n+1$.

We now state the main theorems of this paper. In all the following results, M and M' are assumed to be smooth hypersurfaces in $\mathbf{C}^{n+1}$ and H a germ at p_0 of a smooth CR map from M into M'.

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Theorem 1. *If M is minimal at p_0 and M' does not contain any germ of a nontrivial complex holomorphic manifold in a neighborhood of $H(p_0)$, then either H is constant or the generic rank of H is odd.*

If M is a hypersurface in $\mathbf{C}^{n+1}$, $p_0 \in M$, we say there is a *formal holomorphic vector field at p_0 tangent to M* if there exist $n+1$ formal power series in $n+1$ indeterminates $a_1(Z), \ldots, a_{n+1}(Z)$, not all vanishing at 0, such that if $\rho(Z, \bar{Z})$ is a real-valued smooth defining function for M near p_0 with nonzero differential, then

$$\sum_{1}^{n+1} a_j(Z) \frac{\partial \rho}{\partial Z_j}(Z + p_0, \bar{Z} + \overline{p}_0) = c(Z, \bar{Z}) \rho(Z + p_0, \bar{Z} + \overline{p}_0) \tag{0.1}$$

where $c(Z, \bar{Z})$ is a formal power series in $2n+2$ indeterminates and equality holds in the sense of Taylor series. (For more details, see Section 2.)

Theorem 2. *If M is minimal at p_0 and there is no formal holomorphic vector field at any point in a neighborhood of p_0 tangent to M, then either the generic rank of H is even or is $2n+1$, i.e., $Jac(H) \not\equiv 0$.*

The following is a consequence of Theorems 1 and 2.

Corollary 0.2. *If M is minimal at p_0 with no formal holomorphic vector field at any point in a neighborhood of p_0 tangent to M, and if M' does not contain any germ of a nontrivial holomorphic manifold in a neighborhood of $H(p_0)$, then either H is constant or $Jac(H) \not\equiv 0$.*

It is proved (Lemma 2.5 below) that all essentially finite hypersurfaces satisfy the assumptions of Theorem 2. (See [4] or Section 2 for precise definitions.) Hence we obtain the following.

Theorem 3. *If M is essentially finite at p_0 and M' does not contain any germ of a nontrivial complex holomorphic manifold in a neighborhood of $H(p_0)$, then either H is constant or $Jac(H) \not\equiv 0$.*

Note that any hypersurface of D-finite type at p_0 is also essentially finite [2] and does not contain holomorphic varieties through any point in a neighborhood of p_0 [1]. Hence using Theorem 3 in conjunction with the results of [3], we have:

Corollary 0.3. *Let M and M' be real analytic hypersurfaces in $\mathbf{C}^{n+1}$ and H a smooth CR map from M into M'. If M is essentially finite at every point and M' is of D-finite type at every point, then there exists a dense open set $U \subset M$ such that for every $p \in U$, H extends holomorphically in a neighborhood of p in $\mathbf{C}^{n+1}$.*

The authors do not know if, under the hypotheses of Corollary 0.3, the mapping H extends holomorphically at all points in M.

Another corollary of Theorem 3 is the following.

Corollary 0.4. *If M and M' are of D-finite type at p_0 and $H(p_0)$ respectively, then either H is constant or $Jac(H) \not\equiv 0$.*

Further results relating the generic rank and the dimension of the possible complex manifolds contained in M' as well as the dimension of the vector fields satisfying (0.1) can be found in Section 4 (Theorems $1'$ and $2'$). In [7], Stanton considered hypersurfaces with no tangent holomorphic vector field. It would be interesting to know how our formal condition introduced in (0.1) is related to hers.

The present paper generalizes to the smooth case some aspects of previous work of the authors [6] concerning holomorphic mappings of essentially finite real analytic hypersurfaces in complex space.

1. Preliminaries: Proof of Theorem 1

Assume that near p_0, M is given by $\rho(Z, \bar{Z}) = 0$, where ρ is a smooth, real-valued function vanishing at p_0 and with nonvanishing differential. If $\mathcal{O}$ is a sufficiently small neighborhood of p_0 in $\mathbf{C}^{n+1}$, we denote by $\mathcal{O}^+$ (resp. $\mathcal{O}^-$) the open subset of $\mathcal{O}$ where $\rho(Z, \bar{Z}) > 0$ (resp. $\rho(Z, \bar{Z}) < 0$). Since M is minimal at p_0, H extends holomorphically to one side of M, say $\mathcal{O}^+$, if $\mathcal{O}$ is sufficiently small. We denote this extension by $\mathcal{H}$, so that $\mathcal{H} : \mathcal{O}^+ \to \mathbf{C}^{n+1}$ is holomorphic and $\mathcal{H} \in C^\infty(\overline{\mathcal{U}})$, where $\mathcal{U} = \mathcal{O}^+ \cup (M \cap \mathcal{O})$. We denote $M \cap \mathcal{O}$ by $\mathcal{O}_M$. We shall always assume that $\mathcal{O}$, $\mathcal{O}^+$, $\mathcal{O}_M$, and $\mathcal{U}$ are connected. For $Z \in \mathcal{U}$, we define $\mathrm{rk}_{\mathbf{C}}\mathcal{H}'(Z)$ as the rank of the $(n+1) \times (n+1)$ complex Jacobian matrix $((\partial\mathcal{H}_j/\partial Z_k)(Z))_{1\leq j,k\leq n+1}$, where $\mathcal{H} = (\mathcal{H}_1, \ldots, \mathcal{H}_{n+1})$. Hence for $Z \in \mathcal{O}_M$ we have two ranks: $\mathrm{rk}_M H'(Z)$, the rank of H considered as a map from the real manifold M to the real manifold M' as in the Introduction, and $\mathrm{rk}_{\mathbf{C}}\mathcal{H}'(Z)$ introduced above. We have the following relation between these two:

Lemma 1.1. *For $Z \in \mathcal{O}_M$, where $\mathcal{O}$ is as above, we have*

$$\mathrm{rk}_{\mathbf{C}}\mathcal{H}'(Z) = \left[\frac{\mathrm{rk}_M H'(Z) + 1}{2}\right], \tag{1.2}$$

where $[k]$ denotes the greatest integer less than or equal to k. Furthermore, the maximal rank of $\mathcal{H}$ is achieved on M, and if H has maximal rank at $Z_0 \in \mathcal{O}_M$, then $\mathcal{H}$ also has maximal rank *at Z_0 as a mapping in $\mathcal{U}$. More precisely, we have the inclusion*

$$\{Z \in \mathcal{O}_M : \mathrm{rk}_M H'(Z) \text{ maximal in } \mathcal{O}_M\} \subset \{Z \in \mathcal{O}_M : \mathrm{rk}_{\mathbf{C}} \mathcal{H}'(Z) \text{ maximal in } \mathcal{U}\}. \quad (1.3)$$

In addition, the open set defined by the left-hand side of (1.3) is dense in $\mathcal{O}_M$.

Proof. For (1.2) it suffices to consider the question of ranks in the underlying real Euclidean space $\mathbf{R}^{2n+2}$, where the fact that $\mathcal{H}$ is holomorphic implies that its real rank, $\mathrm{rk}_{\mathbf{R}}\mathcal{H}'(Z)$, is even and $\mathrm{rk}_{\mathbf{R}}\mathcal{H}'(Z) = 2\ \mathrm{rk}_{\mathbf{C}}\mathcal{H}'(Z)$. After local smooth (not necessarily holomorphic) changes of variables in the source and in the target, we can assume that M and M' are the hyperplanes $x_1 = 0$ and $\mathcal{H}$ is defined for $x_1 > 0$ near 0 with $H = \mathcal{H}|_M$. Clearly, for all $p \in M$, we have $\mathrm{rk}_M H'(p) \leq \mathrm{rk}_{\mathbf{R}}\mathcal{H}'(p) \leq \mathrm{rk}_M H'(p) + 1$. Since $\mathrm{rk}_{\mathbf{R}}\mathcal{H}'(p)$ is even, (1.2) follows by considering separately the cases where $\mathrm{rk}_M H'(p)$ is odd and $\mathrm{rk}_M H'(p)$ is even.

For the rest of the lemma, we first observe that since $\mathcal{H}$ is holomorphic in $\mathcal{O}^+$, which is connected, $\{Z \in \mathcal{O}^+ : \mathrm{rk}_{\mathbf{C}}\mathcal{H}'(Z) \text{ maximal in } \mathcal{O}^+\}$ is given by the nonvanishing of a cofactor of the Jacobian determinant of $\mathcal{H}'(Z)$. Hence the restriction of this holomorphic function to $\mathcal{O}$ is nonvanishing in an open dense subset of $\mathcal{O}_M$. This shows that the maximum of $\mathrm{rk}_{\mathbf{C}}\mathcal{H}'(Z)$ for $Z \in \mathcal{U}$ is achieved on M, i.e., the set defined on the right-hand side of (1.3) is nonempty. Now the inclusion (1.3) follows from (1.2). ☐

Let $\mathcal{O}$ be a fixed, sufficiently small open neighborhood of p_0 and r the maximal rank of H in $\mathcal{O}_M$. For the proof of Theorem 1, we shall assume H is not constant, i.e., $r > 0$. By the last part of Lemma 1.1, we may choose $p_1 \in \mathcal{O}_M$ arbitrarily close to p_0 such that $\mathrm{rk}_M H'(p_1) = r$. We shall show that if r is even, i.e., $r = 2k$ with $k > 0$, then M' contains a k-dimensional complex manifold through a point arbitrarily close to $H(p_1)$. We state the following elementary lemma which shows that $\mathcal{H}$ can be extended smoothly to a full neighborhood of p_1 in $\mathbf{C}^{n+1}$ without increasing its rank. We leave the proof of the lemma to the reader.

Lemma 1.4. *Let $\mathcal{O}$ be a neighborhood of 0 in $\mathbf{R}^N$, with $x = (x_1, \ldots, x_N)$ denoting the variable, and let $\mathcal{U} = \mathcal{O} \cap \{x_N \geq 0\}$. Let $\mathcal{J} : \mathcal{U} \to \mathbf{R}^N$ be a smooth map up to the boundary and J be its restriction to $U = \mathcal{O} \cap \{x_N = 0\}$. Suppose that $rk_{\mathbf{R}}\mathcal{J}'(0)$ is maximal in $\mathcal{U}$ and $rk_{\mathbf{R}} J'(0)$ is maximal in U. Then there exists Ω, an open neighborhood of 0 in $\mathbf{R}^N$, and a smooth map $\tilde{\mathcal{J}} : \Omega \to \mathbf{R}^N$ whose restriction to $\mathcal{U} \cap \Omega$ coincides with $\mathcal{J}$ such that*

$$\mathrm{rk}_{\mathbf{R}}\tilde{\mathcal{J}}'(p) = \mathrm{rk}_{\mathbf{R}}\mathcal{J}'(0), \quad \text{for all } p \in \Omega. \quad (1.5)$$

Let $\tilde{\mathcal{H}}$ be a smooth extension of $\mathcal{H}$ to a full neighborhood Ω of p_1 in $\mathbf{C}^{n+1}$ with maximal rank at p_1. Such an extension is given by Lemma 1.4 after a local smooth (nonholomorphic) flattening of M. Suppose, by contradiction, that the generic rank of H is $2k$ with $k > 0$. Note that, using (1.2), we have $\mathrm{rk}_{\mathbf{C}}\mathcal{H}'(p_1) = k$ and $\mathrm{rk}_M H'(p_1) = 2k$. Hence $\mathrm{rk}_{\mathbf{R}}\mathcal{H}(p_1) = \mathrm{rk}_M H'(p_1)$, which implies, again by the implicit function theorem after shrinking Ω if necessary, that $\tilde{\mathcal{H}}(\Omega) = H(\Omega \cap M)$. Hence if $V \subset \Omega \cap \mathcal{O}^+$ is open and arbitrarily close to p_1, then $\mathcal{H}(V)$

is a complex manifold contained in M' arbitrarily close to $H(p_1)$, contradicting the hypothesis on M'. This proves that the generic rank of H is odd, which completes the proof of Theorem 1. □

2. Formal hypersurfaces and formal maps

The notation and results of this section are used in the proof of Theorems 2 and 3. We consider formal power series $\rho(Z,\zeta)$ in $2n+2$ indeterminates, $Z=(Z_1,\ldots,Z_n)$, $\zeta=(\zeta_1,\ldots,\zeta_n)$, i.e., $\rho(Z,\zeta)\in \mathbb{C}[[Z,\zeta]]$. We assume that $\rho(Z,\zeta)$ is *formally real*, that is, if $\rho(Z,\zeta)=\sum \rho_{\alpha\beta}Z^\alpha\zeta^\beta$ then $\rho_{\alpha\beta}=\bar\rho_{\beta\alpha}$, and that $\rho(0,0)=0$, $d\rho(0)\neq 0$. If $\tilde\rho(Z,\zeta)$ is another such formal series, we shall say that $\tilde\rho(Z,\zeta)$ is *equivalent* to $\rho(Z,\zeta)$ if there exists a formally real formal series $b(Z,\zeta)\in\mathbb{C}[[Z,\zeta]]$, with $b(0)\neq 0$, such that

$$\rho(Z,\zeta)=b(Z,\zeta)\tilde\rho(Z,\zeta). \tag{2.1}$$

By a *formal hypersurface* $\mathcal{M}$ at the origin in $\mathbb{C}^{n+1}$ we shall mean an equivalence class of such formal power series. A representative in an equivalence class is called a *formal defining function* for $\mathcal{M}$.

Given $n+1$ formal holomorphic power series, with no constant terms, in $n+1$ indeterminates $Z=(Z_1,\ldots,Z_{n+1})$ denoted $\tilde Z(Z)=(\tilde Z_1(Z),\ldots,\tilde Z_{n+1}(Z))$ and satisfying $\det((\partial\tilde Z_j/\partial Z_k)(0))\neq 0$, we write $\tilde\zeta(\zeta)=(\tilde\zeta_1(\zeta),\ldots,\tilde\zeta_{n+1}(\zeta))$, with $\tilde\zeta_j(\zeta)=\overline{\tilde Z_j}(\zeta)$, where $\overline{\tilde Z_j}(\zeta)$ is obtained from $\tilde Z_j(\zeta)$ by taking complex conjugates of the coefficients. We shall refer to $(\tilde Z(Z),\tilde\zeta(\zeta))$ as a *formal holomorphic change of variables* in $\mathbb{C}[[Z,\zeta]]$.

Let $\mathcal{M}$ be a formal hypersurface in $\mathbb{C}^{n+1}$. We shall call a choice of formal coordinates $Z=(Z_1,\ldots,Z_{n+1})$ *normal* (for $\mathcal{M}$) it there is a formal defining function $\rho(Z,\zeta)$ for $\mathcal{M}$ so that

$$\rho(Z,0)=j(Z)Z_{n+1} \tag{2.2}$$

for some $j(Z)\in\mathbb{C}[[Z]]$ with $j(0)\neq 0$. The existence of normal coordinates can be proved by the formal implicit function theorem. If $Z=(Z_1,\ldots,Z_{n+1})$ are normal coordinates for $\mathcal{M}$, we write $z=(Z_1,\ldots Z_n)$, $w=Z_{n+1}$, $\eta=(\zeta_1,\ldots,\zeta_n)$, and $\tau=\zeta_{n+1}$.

To say that there is a formal holomorphic vector field tangent to $\mathcal{M}$ means that there are formal series $a_j(Z)\in\mathbb{C}[[Z]]$, $j=1,\ldots n+1$, not all vanishing at 0 and $k(Z,\zeta)\in\mathbb{C}[[Z,\zeta]]$ such that

$$\left(\sum_{j=1}^{n+1} a_j(Z)\frac{\partial}{\partial Z_j}\right)\rho(Z,\zeta)=k(Z,\zeta)\rho(Z,\zeta). \tag{2.3}$$

Note that if the coordinates Z are normal for $\mathcal{M}$, it follows from (2.2) and (2.3) by taking $\zeta=0$ and $Z_{n+1}=0$ that necessarily $a_{n+1}(Z_1,\ldots,Z_n,0)=0$. We obtain from (2.2) and (2.3) by taking $w = \tau = 0$,

$$\left(\sum_{j=1}^{n} a_j(z,0)\frac{\partial}{\partial z_j}\right)\rho(z,0,\eta,0) = k(z,0,\eta,0)\rho(z,0,\eta,0). \qquad (2.4)$$

Recall from [4] that if $\rho(z,0,\eta,0) = \sum_\alpha \rho_\alpha(z)\eta^\alpha$, then $\mathcal{M}$ is essentially finite if the ideal $\mathcal{I} = (\rho_\alpha(z))$, generated by the $\rho_\alpha(z)$, is of finite codimension in $\mathbf{C}[[z]]$.

Lemma 2.5. *If there is a formal holomorphic vector field tangent to a formal hypersurface $\mathcal{M}$, then $\mathcal{M}$ is not essentially finite.*

Proof. If such a formal vector field exists, so that (2.4) holds, we expand both sides in η to conclude that $\sum_{j=1}^{n} a_j(z,0)(\partial\rho_\alpha/\partial z_j)(z) \in \mathcal{I}$, where $\mathcal{I}$ is the ideal defined above. By the assumed nonvanishing of one of the a_j at 0 (and since $a_{n+1}(z,0) = 0$, as remarked above), we can find a formal holomorphic change of coordinates in the z_j so that the vector field $\sum_{j=1}^{n} a_j(z,0)\,(\partial/\partial z_j)$ becomes $\partial/\partial z_1$. Hence we may assume $\partial\rho_\alpha/\partial z_1 \in \mathcal{I}$ for all multiindices α. This implies, since the ρ_α generate $\mathcal{I}$, that $\mathcal{I}$ is closed under differentiation with respect to z_1, i.e., if $u(z) \in \mathcal{I}$, then $\partial u/\partial z_1(z) \in \mathcal{I}$. We claim that this implies that $\mathcal{I}$ is of infinite codimension in $\mathbf{C}[[z]]$. Indeed, if $\mathcal{I}$ were of finite codimension, then by the Nullstellensatz there would exist a positive integer N such that $z_1^N \in \mathcal{I}$. Differentiating N times, we obtain $1 \in \mathcal{I}$, which is impossible, since it follows from (2.2) that $\rho_\alpha(0) = 0$ for all α, that is, $\mathcal{I}$ is a proper ideal in $\mathbf{C}[[z]]$. □

A *formal holomorphic map* from $\mathbf{C}^{n+1}$ into $\mathbf{C}^{r+1}$ is an $r+1$ tuple $K = (K_1, \ldots, K_{r+1})$, with $K_j(Z) \in \mathbf{C}[[Z]]$, $K_j(0) = 0$. We shall say that K is *of constant rank p* if there exists a $p \times p$ minor in the Jacobian matrix

$$\left(\frac{\partial K_j(Z)}{\partial Z_k}\right)_{\substack{1\le j\le r+1\\ 1\le k\le n+1}}$$

nonvanishing at 0 and such that all $p' \times p'$ minors vanish identically (as power series) for $p' > p$. This condition is invariant by formal holomorphic changes of variables in both the source and the target. Using a formal implicit function theorem, the reader can check that if K is of constant rank p then there exist formal holomorphic changes of variables (in both the source and target spaces) such that in the new variables $K(Z) = (Z_1, \ldots, Z_p, 0, \ldots, 0)$.

We shall say that a formal mapping K from $\mathbf{C}^{n+1}$ into $\mathbf{C}^{r+1}$ maps one formal hypersurface $\mathcal{M}$ in $\mathbf{C}^{n+1}$ to another formal hypersurface $\mathcal{M}'$ in $\mathbf{C}^{r+1}$ if

$$\rho'(K(Z), \bar{K}(\zeta)) = C(Z,\zeta)\rho(Z,\zeta), \qquad (2.6)$$

where ρ and ρ' are formal defining functions for $\mathcal{M}$ and $\mathcal{M}'$ respectively and $\bar{K}$ is the power series obtained from $\bar{K}$ by taking complex conjugates of the coefficients. Note that (2.6) is independent of both the choice of defining functions and formal holomorphic coordinates.

3. Proof of Theorem 2

Let M, M', and H satisfy the hypotheses of Theorem 2. We shall prove by contradiction that if $Jac\ H \equiv 0$ in a neighborhood of p_0 and the generic rank of H in that neighborhood is odd, say, $2r+1$, with $0 \le r \le n-1$, then in any neighborhood U_0 of p_0 in M there is a formal holomorphic vector field at some $p \in U_0$ tangent to M. Given such a neighborhood U_0, let $p_1 \in U_0$ be such that $\mathrm{rk}_M H'(p_1)$ is maximal in U_0, where we have used the notation introduced in Section 1. For simplicity we may assume $p_1 = 0$. We may choose an open set $U \subset U_0$, with $p_1 \in U$, such that H is of constant rank, $2r+1$, in U. If $\mathcal{O}$ is a sufficiently small neighborhood of 0 in $\mathbf{C}^{n+1}$, as in Section 1, we may assume that H extends holomorphically to $\mathcal{O}^+$; we may also assume $U = \mathcal{O} \cap M$. Writing $\mathcal{U} = \mathcal{O}^+ \cup U$ as before, we obtain from Lemma 1.1, after shrinking $\mathcal{O}$ if necessary, that $\mathrm{rk}_{\mathbf{C}} \mathcal{H}'(Z) = r+1$ for all $Z \in \mathcal{U}$. By Lemma 1.4, we may extend $\mathcal{H}$ smoothly to Ω, a full neighborhood of 0 in $\mathbf{C}^{n+1}$, such that $\mathrm{rk}_{\mathbf{R}} \tilde{\mathcal{H}}(p) = 2r+2$ for all $p \in \Omega$, where $\tilde{\mathcal{H}}$ denotes the smooth extension of $\mathcal{H}$ to Ω. By the implicit function theorem, $H(M \cap \Omega)$ is a manifold of codimension 1 in the $(2r+2)$-dimensional manifold $\tilde{\mathcal{H}}(\Omega)$. Let $\tilde{\mathcal{H}}(\Omega)$ be given near the origin by

$$\{q_j(Z', \bar{Z}') = 0, \qquad j = 1, \ldots, 2n-2r\}, \tag{3.1}$$

with $q_j(Z', \bar{Z}')$ smooth, real-valued functions with linearly independent differentials at 0. Then $H(M \cap \Omega)$ is defined near the origin by (3.1) and the vanishing of an additional real-valued function

$$q_{2n-2r+1}(Z', \bar{Z}') = 0 \tag{3.2}$$

with $dq_1, \ldots, dq_{2n-2r+1}$ linearly independent at 0.

We denote by $K(Z)$ the Taylor series of $\tilde{\mathcal{H}}$ at 0, which is formally holomorphic. Let ρ and ρ' be smooth defining functions of M and M', respectively, in a neighborhood of 0 in $\mathbf{C}^{n+1}$; we continue to denote by $\rho(Z, \bar{Z})$ and $\rho'(Z', \bar{Z}')$ their Taylor series at 0. These formal series define formal hypersurfaces in the sense of Section 2, which we denote by $\mathcal{M}$ and $\mathcal{M}'$ respectively. The formal holomorphic mapping $K(Z)$ maps $\mathcal{M}$ into $\mathcal{M}'$ in the sense of (2.6). Since $\tilde{\mathcal{H}}$ is of real constant rank $2r+2$ in Ω, it is easy to check that the formal holomorphic map $K(Z)$ is of constant rank $r+1$ as defined in Section 2. Hence there are formal holomorphic changes of coordinates both in the source and the target, so that in the new formal coordinates we have $K(Z) = (Z_1, \ldots, Z_{r+1}, 0, \ldots, 0)$. The Taylor series corresponding to (3.1) then becomes $Z'_j = 0, r+2 \le j \le n+1$ and the Taylor series corresponding to (3.2) becomes

$$\rho''(Z'_1, \ldots, Z'_{r+1}, \zeta'_1, \ldots, \zeta'_{r+1}) = 0, \text{ with } d\rho''(0) \neq 0,$$

where ρ'' is a formally real formal power series in $2r+2$ indeterminates vanishing at 0. Let $\mathcal{M}''$ be the formal hypersurface in $\mathbf{C}^{r+1}$ defined by ρ''. Since $H(M) \subset M'$, by composition of

Taylor series we obtain that K maps $\mathcal{M}$ into $\mathcal{M}''$ in the sense of (2.6), i.e.,

$$\rho''(Z_1, \ldots, Z_{r+1}, \zeta_1, \ldots, \zeta_{r+1}) = C(Z, \zeta)\rho(Z, \zeta). \tag{3.3}$$

Since $\rho(0) = 0$, the nonvanishing of $d\rho''(0)$ implies $C(0) \neq 0$. Hence the left-hand side of (3.3) is a formal defining function for $\mathcal{M}$ that is independent of $Z_{r+2}, \ldots, Z_{n+1}$, so that the formal holomorphic vector fields $\partial/\partial Z_{r+2}, \ldots, \partial/\partial Z_{n+1}$ are all tangent to $\mathcal{M}$, contradicting the assumption on M. This completes the proof of Theorem 2. □

4. Remarks and further results

If M and M' are smooth hypersurfaces in $\mathbf{C}^{n+1}$ and H a germ at p_0 of a CR map from M into M' whose rank is constant in a small neighborhood U of p_0 in M, then it can be easily shown that $H(U)$ is a CR manifold in $\mathbf{C}^{n+1}$. In addition, it follows from a slight modification of Lemma (1.1) that the CR dimension of $H(U)$ is $[\mathrm{rk}_M H'(p)/2]$ for $p \in U$. The previous remark, and an inspection of the proof of Theorem 1, yields the following generalization.

Theorem 1′. *If the rank of H is constant in a neighborhood of p_0 in M and M' does not contain any germ of a holomorphic manifold of complex dimension $r > 0$ through $H(p_0)$, then the rank of H in that neighborhood is either odd or less than $2r$.*

To obtain a similar generalization for Theorem 2, we first note that the set of all formal holomorphic vector fields at p_0 tangent to M (i.e., those satisfying (0.1)) forms a Lie algebra. Let $d(p_0)$ be its (complex) dimension at p_0. Note that we always have $0 \leq d(p_0) \leq n$. Then the following result is obtained using the remarks above in conjunction with Theorem 6 in [5] and an inspection of the proof of Theorem 2.

Theorem 2′. *If the rank of H is constant in a neighborhood of p_0 in M and $d(p_0) < r$, then the rank of H in that neighborhood is either even or greater than $2(n - r) + 1$.*

It should be noted that in Theorems 1′ and 2′ the hypersurface M is not assumed to be minimal at p_0.

Remark 4.1. In the case of $\mathbf{C}^2$, i.e., $n = 1$, Theorems 1 and 2 (or 1′ and 2′) yield the following: If M is minimal at p_0 and M' is minimal at every point in a neighborhood of $H(p_0)$, then either H is constant or the generic rank of H is three. This conclusion fails if we assume only that M' is minimal at $H(p_0)$, as is shown by Example 4.2 below. □

Example 4.2. Let M be the Lewy hypersurface in $\mathbf{C}^2$ given by $\operatorname{Im} w = |z|^2$. Let $\phi(t) = e^{-1/t^2}$ for $t < 0$ and 0 otherwise, and M' the hypersurface in $\mathbf{C}^2$ given by $\operatorname{Im} w' = \phi(y')$, where $z' = x' + iy'$. It is easy to check that M' is minimal at 0, but not at any point where

$y' > 0$. The holomorphic mapping $\mathcal{H}(z,w) = (w,0)$ restricted to M is a CR map from M into M' with generic rank equal to two. $\square$

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