SEMI-RIGID CR STRUCTURES
AND HOLOMORPHIC EXTENDABILITY

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Let $\Omega \subset \mathbb{R}^{2n+\ell}$ be an open set, $0 \in \Omega$, and $\mathcal{T}\Omega$, the complexified tangent bundle to $\Omega$. Let $\mathcal{V}$ be a subbundle of $\mathcal{T}\Omega$ such $\dim_{\mathbb{C}} \mathcal{V}_\omega = n$, $\forall \omega \in \Omega$. We denote by $\mathcal{L}$ the space of smooth sections of $\mathcal{V}$ defined in $\Omega$. We shall assume the Frobenius condition, i.e.

$$[\mathcal{V}, \mathcal{V}] \subset \mathcal{V},$$

and also

$$\mathcal{V}_\omega \cap \overline{\mathcal{V}}_\omega = \{0\}, \forall \omega \in \Omega.$$

With the above assumptions we say that $\Omega$ is equipped with an abstract CR structure of codimension $\ell$.

If in addition for every $\omega_0 \in \Omega$, there exist an open set $\Omega' \subset \Omega$, $\omega_0 \in \Omega'$, and smooth functions in $\Omega'$, with independent differentials, $Z_1, \ldots, Z_{n+\ell}$, satisfying

$$L Z_j = 0, \ j = 1, \ldots, n+\ell, \ \forall L \in \mathcal{L},$$

we say that $\mathcal{V}$ (or $\mathcal{L}$) is locally integrable. We denote by $\mathcal{M} \subset \mathbb{C}^{n+\ell}$ the image of $\Omega'$. It is a (germ of a) generic CR manifold of codimension $\ell$.

We shall say that $\mathcal{V}$ is of finite type in $\Omega$ at $\omega$ (see Kohn [9] or Bloom-Graham [5] if for any $\xi \in T^*_{\omega} \Omega \setminus \{0\}$ there exists a commutator.
\( L^{(k)} = \{ L_1, L_2, \ldots, L_{k-1}, L_k \} \),
each \( L_j \in \mathcal{L} \otimes \mathcal{L} \), such that the symbol \( \sigma(L^{(k)}) \) satisfies

\[ \sigma(L^{(k)})(\omega, \xi) \neq 0. \]

Let \( m(\omega, \xi) \) be the smallest integer \( k \) such that (2) is satisfied. The Hörmander numbers at \( \omega \) are the \( r \) distinct integers \( 2 \leq m_1 < m_2 \ldots < m_r \) obtained as \( m(\omega, \xi) \) for some \( \xi \in T_{\omega}^* \Omega \setminus \{0\} \), \( \xi \) characteristic for \( \mathcal{L} \).

We shall say that a CR structure \( V \) of finite type is semi-rigid at \( \omega \) if for all \( \xi \in T_{\omega}^* \Omega \)

\[ \sigma([L^{(k)}, L^{(p)}])(\omega, \xi) = 0 \]

for all commutators \( L^{(k)}, L^{(p)} \) of the form (1) with \( k, p \geq 2 \) and \( k + p \leq m(\omega, \xi) \).

The associated embedded generic CR manifold \( \mathcal{M} \) will also be said to be semi-rigid.

The following result gives local normal forms for such manifolds.

**Theorem 1:** Let \( \mathcal{M} \) be a generic CR manifold of codimension \( \ell \) in \( \mathbb{C}^{n+\ell} \).

If \( \mathcal{M} \) is of finite type at the origin, there are holomorphic coordinates around the origin, \( (z, \omega) \in \mathbb{C}^{n+\ell} \) such that on \( \mathcal{M} \)

\[ z_j = x_j + iy_j, \quad 1 \leq j \leq n, \]

\[ w_k = s_k + i \left[ p_{m_k} (z, \bar{z}, s_1, \ldots, s_{k-1}) + O(m_k + 1) \right], \quad 1 \leq k \leq r, \]

where \( p_{m_k} \) is homogeneous of weight \( m_k \) and \( O(m_k + 1) \) is of weight \( m_k + 1 \). Here the \( x, y \in \mathbb{R}^n \) are given weight 1, while \( s_j \in \mathbb{R}^{\ell_j} \) is given weight \( m_j \), and \( \ell_1 + \ldots + \ell_r = \ell \). Furthermore, the \( p_{m_k} \) may be chosen independent of all the \( s_j \) if and only if \( \mathcal{M} \) is semi-rigid.

The first statement of Theorem 1 is in Bloom-Graham [5]; our proof, as well as the proof of the second statement, uses methods of Helffer-Nourrigat [7].
The following are examples of semi-rigid CR manifolds:

1. Any hypersurface in $\mathbb{C}^{n+1}$ of finite type.
2. Any generic CR manifold of finite type in $\mathbb{C}^{n+2}$ with Hörmander's numbers $m_j \leq 3$, for all $j$.
3. Any generic CR manifold of finite type such that there exists $m \geq 2$ satisfying $m \leq m_j \leq m+1$ for all $j$.

A function $h$ on $M$ is said to be CR if it satisfies the equations

$$Lh = 0 \quad \text{for all } L \in \mathbb{L}.$$ 

We are concerned with the holomorphic extendability of CR functions across a point in $M$.

In order to state our main result we shall define the following sets of extendability. If a generic CR manifold in $\mathbb{C}^{n+2}$ is defined by

$$\text{Im } w = \phi(z, \overline{z}, \text{Re } w), \quad z \in \mathbb{C}^n, \quad w \in \mathbb{C}^2,$$

$\phi(0) = 0$, $\phi'(0) = 0$, and if $\Gamma$ is a strictly convex open cone in $\mathbb{R}^2 \setminus \{0\}$, a wedge with edge $M$ is defined by

$$\Omega_\Gamma = \{ (z, w) \in 0 \subset \mathbb{C}^{n+2} : \text{Im } w - \phi(z, \overline{z}, \text{Re } w) \in \Gamma \},$$

where $0$ is a neighborhood of $0$.

**Theorem 2.** Let $M$ be a semi-rigid CR manifold of finite type at the origin. Then any CR function on $M$ extends holomorphically to a wedge of the form (5).

When the CR manifold $M$ defined by (4) is real analytic, we have the following nonextendability result:

**Theorem 3.** Assume that $M$ is a generic real analytic CR manifold in $\mathbb{C}^{n+2}$ which is not of finite type at the origin. Then there exists a CR function defined near $0$ on $M$ which does not extend to any wedge.

Many extendability results have been proved since the classical work of H. Lewy [8]. Some recent ones are [3], [6], [4], [11]. A weaker version of Theorem 2 is proved in [2].

As in [2], the proof of Theorem 2 is based on the use of a generalized FBI transform (see Sjöstrand [10] and [11]) of the form

$$\int e^{i(w-s-\phi(z, \overline{z}, s))} - |\sigma| (w-s-\phi(z, \overline{z}, s))^2 \chi(s)h(x,y,s)\det(I+i\phi_s(z, \overline{z}, s))ds.$$

Details of proofs will appear elsewhere.
Références :


