SEMI-RIGID CR STRUCTURES AND HOLOMORPHIC EXTENDABILITY

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Let $\Omega \subset \mathbb{R}^{2n+\ell}$ be an open set, $0 \in \Omega$, and $\mathbb{C}T\Omega$, the complexified tangent bundle to Ω. Let $\mathcal{V}$ be a subbundle of $\mathbb{C}T\Omega$ such $\dim_{\mathbb{C}} \mathcal{V}_\omega = n$, $\forall\ \omega \in \Omega$. We denote by $\mathbb{L}$ the space of smooth sections of $\mathcal{V}$ defined in Ω. We shall assume the Frobenius condition, i.e.

$$[\mathcal{V},\mathcal{V}] \subset \mathcal{V} \quad ,$$

and also

$$\mathcal{V}_\omega \cap \overline{\mathcal{V}_\omega} = \{0\} \quad , \quad \forall\ \omega \in \Omega \ .$$

With the above assumptions we say that Ω is equipped with an <u>abstract</u> CR <u>structure</u> of codimension ℓ.

If in addition for every $\omega_o \in \Omega$, there exist an open set $\Omega' \subset \Omega$, $\omega_o \in \Omega'$, and smooth functions in Ω', with independent differentials, $Z_1,\ldots,Z_{n+\ell}$, satisfying

$$L\,Z_j = 0 \ , \ j = 1,\ldots,n+\ell \ , \quad \forall\ L \in \mathbb{L} \ ,$$

we say that $\mathcal{V}$ (or $\mathbb{L}$) is <u>locally integrable.</u> We denote by $M \subset \mathbb{C}^{n+\ell}$ the image of Ω'. It is a (germ of a) generic CR manifold of codimension ℓ.

We shall say that $\mathcal{V}$ is of finite type in Ω at ω (see Kohn [9] or Bloom-Graham [5] if for any $\xi \in T^*_\omega\Omega \setminus \{0\}$ there exists a commutator

(1) $$L^{(k)} = [\overset{(-)}{L}_1, [\overset{(-)}{L}_2, \ldots, [\overset{(-)}{L}_{k-1}, \overset{(-)}{L}_k] \ldots]] \quad ,$$

each $\overset{(-)}{L}_j \in \mathbb{L} \oplus \overline{\mathbb{L}}$, such that the symbol $\sigma(L^{(k)})$ satisfies

(2) $$\sigma(L^{(k)})(\omega,\xi) \neq 0 \ .$$

Let $m(\omega,\xi)$ be the smallest integer k such that (2) is satisfied. The <u>Hörmander numbers</u> at ω are the r distinct integers $2 \leqslant m_1 < m_2 \ldots < m_r$ obtained as $m(\omega,\xi)$ for some $\xi \in T^*_\omega \Omega \setminus \{0\}$, ξ characteristic for $\mathbb{L}$.

We shall say that a CR structure V of finite type is <u>semi-rigid</u> at ω_o if for all $\xi \in T_{\omega_o} \Omega$

$$\sigma([L^{(k)}, L^{(p)}])(\omega_o,\xi) = 0$$

for all commutators $L^{(k)}$, $L^{(p)}$ of the form (1) with $k,p \geqslant 2$ and $k + p \leqslant m(\omega_o,\xi)$. The associated embedded generic CR manifold M will also be said to be <u>semi-rigid</u>.

The following result gives local normal forms for such manifolds.

<u>Theorem 1</u>: <u>Let</u> M <u>be a generic</u> CR <u>manifold of codimension</u> ℓ <u>in</u> $\mathbb{C}^{n+\ell}$. <u>If</u> M <u>is of finite type at the origin</u>, <u>there are holomorphic coordinates around the origin</u>, $(z,w) \in \mathbb{C}^{n+\ell}$ <u>such that on</u> M

$$z_j = x_j + i\, y_j \qquad 1 \leqslant j \leqslant n \ ,$$

$$w_k = s_k + i\,[\, p_{m_k}(z,\overline{z}, s_1,\ldots,s_{k-1}) + O(m_k + 1)\,] \qquad 1 \leqslant k \leqslant r \ ,$$

<u>where</u> p_{m_k} <u>is homogeneous of weight</u> m_k <u>and</u> $O(m_k + 1)$ <u>is of weight</u> $m_k + 1$. <u>Here the</u> $x,y \in \mathbb{R}^n$ <u>are given weight</u> 1, <u>while</u> $s_j \in \mathbb{R}^{\ell_j}$ <u>is given weight</u> m_j, <u>and</u> $\ell_1 + \ldots + \ell_r = \ell$. <u>Furthermore, the</u> p_{m_k} <u>may be chosen independent of all the</u> s_j <u>if and only if</u> M <u>is semi-rigid</u>.

The first statement of Theorem 1 is in Bloom-Graham [5] ; our proof, as well as the proof of the second statement, uses methods of Helffer-Nourrigat [7].

The following are examples of semi-rigid CR manifolds :

1 - Any hypersurface in $\mathbb{C}^{n+1}$ of finite type.

2 - Any generic CR manifold of finite type in $\mathbb{C}^{n+\ell}$ with Hörmander's numbers $m_j \leq 3$, for all j.

3 - Any generic CR manifold of finite type such that there exists $m \geq 2$ satisfying $m \leq m_j \leq m+1$ for all j.

A function h on M is said to be CR if it satisfies the equations

$$Lh = 0 \quad \text{for all} \quad L \in \mathbb{L} .$$

We are concerned with the holomorphic extendability of CR functions across a point in M.

In order to state our main result we shall define the following sets of extendability. If a generic CR manifold in $\mathbb{C}^{n+\ell}$ is defined by

$$\text{(4)} \qquad \operatorname{Im} w = \Phi(z,\bar{z},\operatorname{Re} w), \quad z \in \mathbb{C}^n, \quad w \in \mathbb{C}^\ell,$$

$\Phi(0) = 0$, $\Phi'(0) = 0$, and if Γ is a strictly convex open cone in $\mathbb{R}^\ell \setminus \{0\}$, a <u>wedge</u> with edge M is defined by

$$\text{(5)} \qquad \mathcal{W}_\Gamma = \{(z,w) \in \mathcal{O} \subset \mathbb{C}^{n+\ell} : \operatorname{Im} w - \Phi(z,\bar{z},\operatorname{Re} w) \in \Gamma\},$$

where $\mathcal{O}$ is a neighborhoood of 0.

<u>Theorem 2</u>. <u>Let</u> M <u>be a semi-rigid</u> CR <u>manifold of finite type at the origin</u>. <u>Then any</u> CR <u>function on</u> M <u>extends holomorphically to a wedge of the form</u> (5).

When the CR manifold M defined by (4) is real analytic, we have the following nonextendability result :

<u>Theorem 3</u>. <u>Assume that</u> M <u>is a generic real analytic</u> CR <u>manifold in</u> $\mathbb{C}^{n+\ell}$ <u>which is not of finite type at the origin</u>. <u>Then there exists a</u> CR <u>function defined near</u> 0 <u>on</u> M <u>which does not extend to any wedge</u>.

Many extendability results have been proved since the classical work of H. Lewy [8]. Some recent ones are [3], [6], [4], [11]. A weaker version of Theorem 2 is proved in [2].

As in [2], the proof of Theorem 2 is based on the use of a generalized FBI transform (see Sjöstrand [10] and [1]) of the form

$$\int e^{i(w-s-i\,\Phi(z,\bar{z},s))\cdot\sigma - |\sigma|(w-s-i\,\Phi(z,\bar{z},s))^2} \chi(s) h(x,y,s) \det(I + i\,\Phi_s(z,\bar{z},s))\,ds .$$

Details of proofs will appear elsewhere.

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