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Inventiones mathematicae

Smoothness and analyticity for solutions of first order systems of partial differential equations on nilpotent Lie groups

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§ 1. Introduction

Let $\mathfrak{g}=\mathfrak{g}_1\oplus\ldots\oplus\mathfrak{g}_r$ be a nilpotent Lie algebra (over $\mathbb{R}$) with $[\mathfrak{g}_i,\mathfrak{g}_j]\subseteq\mathfrak{g}_{i+j}$, and let $G=\operatorname{Exp}\mathfrak{g}$ be the corresponding simply connected group. We assume that $\mathfrak{g}_1$ generates $\mathfrak{g}$, and we write $\mathfrak{g}^j=\sum_{k\geq j}\mathfrak{g}_k$. Let $\mathbb{L}$ be a complex subspace of $\mathfrak{g}_1\otimes\mathbb{C}$. We shall say that $\mathbb{L}$ is *hypoelliptic* (respectively *analytic-hypoelliptic*) if the following statement holds: for each open set $\Omega\subset G$ and each $u\in\mathscr{D}'(G)$,

$$Lu\in C^\infty(\Omega)(\text{resp. } A(\Omega)) \quad \text{for every } L\in\mathbb{L}\Rightarrow u\in C^\infty(\Omega)(\text{resp. } A(\Omega)).$$

In this paper, we shall give necessary and sufficient conditions for hypoellipticity and analytic hypoellipticity.

Our criteria use the following hypotheses on $\mathbb{L}$:

(H1) $\mathbb{L}+\bar{\mathbb{L}}=\mathfrak{g}_1\otimes\mathbb{C}$;

(H2) for every $\lambda\in\mathfrak{g}_2^*\text{-}(0)$ which vanishes on $\operatorname{Re}[\mathbb{L},\mathbb{L}]+\operatorname{Im}[\mathbb{L},\mathbb{L}]$,

the Hermitian form on $\mathbb{L}\times\mathbb{L}$ defined by

$$\langle L,L'\rangle_\lambda=\frac{1}{i}\lambda([L,\bar{L}'])$$

has at least one negative eigenvalue.

Remarks. 1. In fact, $\langle L,L'\rangle_\lambda$ must also have at least one positive eigenvalue (replace λ by $-\lambda$).

2. If $\operatorname{Re}[\mathbb{L},\mathbb{L}]+\operatorname{Im}[\mathbb{L},\mathbb{L}]=\mathfrak{g}_2$, then (H2) is vacuous.

3. The form $\langle\ ,\ \rangle_\lambda$ is the *Levi form* associated to $\mathbb{L}$; it is usually introduced under the further hypothesis that $[\mathbb{L},\mathbb{L}]=(0)$.

Our main results are the following:

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Theorem 1. $\mathbb{L}$ *is hypoelliptic*$\Leftrightarrow$(H1) *and* (H2) *hold for* $\mathbb{L}$.

Theorem 2. *Assume* $[\mathfrak{g}^2, \mathfrak{g}^2]=(0)$. *Then* $\mathbb{L}$ *is analytic-hypoelliptic*$\Leftrightarrow$(H1) *and* (H2) *are satisfied for* $\mathbb{L}$.

In the case $r=2$, these theorems are not new. The C^∞ results follow from more general work of Kohn [14] on the $\bar{\partial}$ Neumann problem on the boundary of a domain in $\mathbb{C}^n$. The analyticity theorem for $r=2$ was proved by Métivier [15], following earlier work (also on the $\bar{\partial}$ Neumann problem) by Treves [18] and Tartakoff [17]. For $r>2$, the criterion of Helffer-Nourrigat [9] gives necessary and sufficient conditions for C^∞ hypoellipticity, but the conditions of our Theorem 1 are simpler.

Sufficient conditions for analytic hypoellipticity for a system of vector fields (not on a group) are given in Baouendi-Treves [3], Baouendi-Chang-Treves [1], and Baouendi-Rothschild [2]. We use these criteria in our proof of Theorem 2. We note that the idea of a criterion based mainly on the Levi form, as well as some of the estimates of Theorem 1, appeared in Derridj [5] (see also Grigis-Rothschild [7] and Helffer-Nourrigat [11]).

The proofs of necessity in Theorems 1 and 2 are more or less known; we give details in §2. The proof of sufficiency depends on the following key algebraic result:

(1.1) **Proposition.** *If* $\mathbb{L}$ *satisfies* (H1) *and* (H2), *then there exists a basis* $L_1, \dots, L_n$ *of* $\mathbb{L}$ *such that*

$$\frac{1}{i}\sum_{j=1}^{n}[L_j, \bar{L}_j] \in \mathrm{Re}[\mathbb{L}, \mathbb{L}] + \mathrm{Im}[\mathbb{L}, \mathbb{L}]. \tag{1.2}$$

This proposition will be proved in Sect. 6. Using it, we reduce the question of C^∞ hypoellipticity to simple L^2 estimates in the spirit of [5] (see [11]). We also use Proposition 1.1 to reduce the question of analytic hypoellipticity to a case where a criterion of [2] can be applied. For the case where $[\mathbb{L}, \mathbb{L}] \neq 0$, we are forced to consider a more general problem out of the group setting. Let $\Omega \subset \mathbb{R}^M$ be open, let $\omega_0 \in \Omega$, let $A(\Omega)$ denote the space of real analytic functions on Ω, and let $\mathscr{L}$ be a module of real analytic vector fields on Ω. We assume that $\mathscr{L}$ satisfies the following conditions near ω_0:

(i) $[\mathscr{L}, \mathscr{L}] \subset \mathscr{L}$.

(ii) $\mathscr{L} \cap \bar{\mathscr{L}} = (0)$.

(iii) For every nonzero $\xi \in T^*_{\omega_0}(\Omega)$, the cotangent space to Ω lying over ω_0, there is an iterated bracket

$$L = [L'_{i_1}, [L'_{i_2}, \dots, [L'_{i_{k-1}}, L'_{i_k}] \dots],$$

with each $L'_{i_j} \in \mathscr{L} + \bar{\mathscr{L}}$, such that the symbol $L(\omega_0, \xi)$ is nonzero.

(iv) For nonzero $\xi \in T^*_{\omega_0}(\Omega)$, let $r(\xi)$ be the smallest possible length of a bracket satisfying (iii). Then for any commutators $L'^{(\alpha)}$, $L'^{(\beta)}$ of $\mathscr{L} + \bar{\mathscr{L}}$ of lengths $|\alpha|$ and $|\beta|$ respectively, the symbol of $[L'^{(\alpha)}, L'^{(\beta)}]$ at (ω_0, ξ) is zero if $|\alpha| + |\beta| \leq r(\xi)$ and $|\alpha|$ and $|\beta|$ are both ≥ 2. Conditions (i) and (ii) imply that $\mathscr{L}$ defines a Cauchy-Riemann (or CR) structure, and condition (iii) is the usual microlocal "Hörmander condition"; see [12]. Condition (iv) is a technical one imposed in [2]; it corresponds to the assumption in Theorem 2 that $[\mathfrak{g}^2, \mathfrak{g}^2] = 0$. (We do not know if this condition is necessary for the conclusion.) We prove Theorem 2 by reducing the problem to one where we can use a slight variation of the following theorem, which seems interesting in its own right:

Theorem 3. *Let $\mathscr{L}$ satisfy* (i)–(iv) *above, and suppose further that $\mathscr{L}$ has a basis* $L_1, \dots, L_n$ (for $\mathscr{L}$ as an $A(\Omega)$-module) *such that*

$$\sum_{j=1}^{n} [L_j, \bar{L}_j] \in \mathscr{L} + \bar{\mathscr{L}}.$$

Then $\mathscr{L}$ is analytic hypoelliptic at ω_0.

§ 2. Proof of Theorem 1

To prove that (H1) and (H2) imply C^∞ hypoellipticity, we use the basis $\{L_j\}$ satisfying (1.2). Then we have complex numbers α_{kl} such that

$$\frac{1}{i} \sum_{j=1} [L_j, \bar{L}_j] = \operatorname{Re} \sum_{k,l} a_{kl} [L_k, L_l]. \tag{2.1}$$

For any $u \in C_0^\infty(G)$ and any $\varepsilon > 0$,

$$|(L_k L_l u, u)| = |(L_l u, \bar{L}_k u)| \leqq \varepsilon \|\bar{L}_k u\|^2 + \varepsilon^{-1} \|L_l u\|^2;$$

hence

$$|([L_k, L_l] u, u)| \leqq \varepsilon (\|\bar{L}_k u\|^2 + \|\bar{L}_l u\|^2) + \varepsilon^{-1} (\|L_k u\|^2 + \|L_l u\|^2),$$

and thus for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that

$$|(\operatorname{Re} \sum_{k,l} \alpha_{kl} [L_k, L_l] u, u)| \leqq \sum_{j=1}^{n} (\varepsilon \|\bar{L}_j u\|^2 + C_\varepsilon \|L_j u\|^2). \tag{2.2}$$

But since $L_j^* = -\bar{L}_j$, we have the following identity:

$$\begin{aligned} \sum_{j=1}^{n} \|L_j u\|^2 &= \sum_{j=1}^{n} (L_j^* L_j u, u) \\ &= \sum_{j=1}^{n} (L_j L_j^* u, u) + \sum_{j=1}^{n} ([L_j^* L_j] u, u) \\ &= \sum_{j=1}^{n} \|\bar{L}_j u\|^2 - i \operatorname{Re} \sum_{k,l} (\alpha_{kl} [L_k, L_l] u, u). \end{aligned} \tag{2.2}$$

Applying (2.1), we get

$$\sum_{j=1}^{n} \|L_j u\|^2 \geqq \sum_{j=1}^{n} \|\bar{L}_j u\|^2 - \sum_{j=1}^{n} (\varepsilon \|\bar{L}_j u\|^2 + C_\varepsilon \|L_j u\|^2);$$

hence (setting $\varepsilon=\frac{1}{2}$),

$$\sum_{j=1}^{n}(\|L_j u\|^2+\|\bar{L}_j u\|^2)\leqq C'\sum_{j=1}^{n}\|L_j u\|^2, \quad C'=3+2C_{\frac{1}{2}}. \tag{2.3}$$

Now the hypoellipticity of $\mathbb{L}$ follows from (2.3) and (H 1), using standard arguments (see, e.g., Kohn [14] or Hörmander [12]).

To show that (H 1) and (H 2) are necessary for hypoellipticity, we use the Rockland criterion: if $\mathbb{L}$ is hypoelliptic, then

$$\bigcap_{L\in\mathbb{L}} \operatorname{Ker}\pi(L)=0 \tag{2.7}$$

for all nontrivial $\pi\in G^\wedge$, where $G^\wedge$ is the set of (equivalence classes of) unitary representations of G. The proofs that this condition is necessary for hypoellipticity of a single left invariant differential operator on G (see 4, 16, 15, 8]) can all be adapted to systems; we sketch the procedure for one such proof. Suppose that π and f are such that $\pi(L)f=0$, all $L\in\mathbb{L}$. For each dilation $x\to\alpha(t)x$ ($t\in\mathbb{R}$) of G, define the representation π_t by $\pi_t(x)=\pi(\alpha_t(x))$. Then $\pi_t(L)f=0$, too. Now let μ be any finite measure on $(0,\infty)$, and define a distribution $\chi(u)$ on G by

$$(\chi(\mu),\varphi)=\int_G(\pi_t(\varphi)f,f)\,d\mu(\pi).$$

Then for any $L\in\mathbb{L}$,

$$\begin{aligned}(L\chi(\mu),\varphi)&=(\chi(\mu),L^t(\varphi))=\int_G\pi_t(L^t(\varphi)f,f)\,d\mu(\pi)\\&=\int_G(\pi_t(\varphi)\pi_t(L)f,f)\,d\mu(\pi)=0,\end{aligned}$$

so that $L\chi(\mu)=0$. One can then choose μ so that $\chi(\mu)\notin C^\infty(G)$; thus $\mathbb{L}$ is not hypoelliptic.

Clearly (H 1) is equivalent to the injectivity of $\pi(\mathbb{L})$ when $\pi\neq 0$ is trivial on $\mathfrak{g}^2$. It is not hard to check that (H 2) is equivalent to the injectivity of $\pi(L)$ for all π trivial on $(\mathfrak{g}^3+\operatorname{Im}[\mathbb{L},\mathbb{L}]+\operatorname{Re}[\mathbb{L},\mathbb{L}])$, but not trivial on $\mathfrak{g}$; the computations are also given in [11] as Lemma V.2.2.7. Note that this proof also shows that (H 1) and (H 2) are necessary for analytic hypoellipticity.

Remarks. 1. The condition (2.7) is also sufficient for C^∞-hypoellipticity for more general systems of left invariant differential operators; see [11]. Here we show that a weaker condition is sufficient for the hypoellipticity of $\mathbb{L}$, since (H 1) and (H 2) say nothing directly about the injectivity of $\pi(\mathbb{L})$ if π is nontrivial on $\operatorname{Exp}(\mathfrak{g}^3+\operatorname{Im}[\mathbb{L},\mathbb{L}]+\operatorname{Re}[\mathbb{L},\mathbb{L}])$.

§ 3. Proof of sufficiency: preliminaries

The proof of sufficiency for Theorem 2 involves three main steps. Let $\mathscr{L}$ be the Lie subalgebra of $\mathfrak{g}\otimes\mathbb{C}$ generated by $\mathbb{L}$. We first prove a "lifting" lemma which lets us reduce to the case where $\mathscr{L}\cap\bar{\mathscr{L}}=(0)$. For this reduction, however, we pay a price: we must work with systems defined on homogeneous spaces. We then show that we can reduce to the case where the principal homogeneous terms of the vector fields satisfy the hypotheses of Theorem 3. Finally, we prove a result which implies both Theorem 3 and Theorem 2. The first step in this program is carried out in §4, and the last two in §5. Here we collect some facts, proved in [2], which will be needed in the next two sections. They come under two general headings: (A) homogeneity, and (B) the sector property.

(A) Homogeneity. Let $\mathscr{L}$ be an $A(\Omega)$-module of real-analytic complex vector fields satisfying (i)–(iv) of §1. As in [2] (see also [10]), we introduce coordinates and a system of dilations. Let $Z_1, \ldots, Z_N$ be any basis of $\mathscr{L}$, and choose commutators of the $\operatorname{Re} Z_j$, $\operatorname{Im} Z_j$, which we denote by S_{jk} (S_{jk} has length j) such that

(1) The fields $\operatorname{Re} Z_j$, $\operatorname{Im} Z_k$, S_{jk} form a basis for the tangent space at ω_0;

(2) for each j_0, any commutator of length $\leqq j_0$ is a linear combination of the vectors $\operatorname{Re} Z_j(\omega_0)$, $\operatorname{Im} Z_k(\omega_0)$, and the $S_{jk}(\omega_0)$ with $j \leqq j_0$ at ω_0.

We call the coordinate system

$$(z, s) \leftrightarrow \exp\left(\sum z_j Z_j\right) \exp(s_{jk} S_{jk}) \cdot \omega_0 \tag{3.1}$$

the *exponential coordinates induced* by $Z_1, \ldots, Z_N$. As in [2] (see also [16], [10]), we introduce the dilations $\delta_t(z, s_{jk}) = (tz, t^j s_{jk})$; they induce a notion of homogeneity for functions and vector fields on Ω in which the Z_k have weight $\geqq -1$, the S_{jk} have weight $\geqq -j$ and a homogeneous polynomial in the z's of degree k has weight k. It is shown in [2] that in the induced exponential coordinates we have

$$Z_j = \frac{\partial}{\partial \bar{z}_j} + i \sum \frac{\partial \varphi_{lk}(z, \bar{z})}{\partial z_j} \frac{\partial}{\partial s_{lk}} + \mathcal{O}, \tag{3.2}$$

where φ_{lk} is a homogeneous polynomial of degree l and $\mathcal{O}$ is of weight $\geqq 0$.

(B) The Sector Property. Let $p = p_m + p_{m-1} + \ldots + p_0$ be a real-valued polynomial on $\mathbb{R}^{2n} \cong \mathbb{C}^n$, with p_j homogeneous of degree j. Following [3], we say that p has the *sector property* if

(3.3) there are pluriharmonic polynomials $q_m^{(j)}$, homogeneous of degree m, and complex lines $l_j \subset \mathbb{C}^n$ ($j = 1, 2$) such that

(3.3a) the restriction of $p_m + q_m^{(1)}$ to l_1 is positive on a sector of angle $> \pi/m$; and

(3.3b) the restriction of $p_m + q_m^{(2)}$ to l_2 is negative on a sector of angle $> \pi/m$.

The following useful criterion for the sector property is an immediate consequence of results in [3]:

Proposition A. *Let* $p(z, \bar{z}) = p_m + \ldots + p_0$ *be a real-valued polynomial on* $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. *Suppose that there exists a complex linear* map $f_1 \colon \mathbb{C} \to \mathbb{C}^n$ *and a number* θ_1 *such that*

(3.4) $$\tilde{p}_m(f_1,\theta_1)=p_m(f_1(e^{i\theta_1}),f_1(\overline{i\theta_1}))+p_m(f_1(e^{i/\theta_1+\pi/m}),f_1(\overline{i(\theta_1+\pi/m)}))$$

is positive. Then (3.3a) *holds for p. If there exist* f_2, θ_2 *such that* $\tilde{p}_m(f_2,\theta_2)<0$, *then* (3.3b) *holds for p.*

Now let $\omega_0 \in \Omega$. A system $\mathscr{P}$ of differential operators on Ω with analytic coefficients is said to be *analytic-hypoelliptic at* ω_0 if for any u such that $\mathscr{P}u$ is analytic in a neighborhood U_1 of ω_0 there exists a neighborhood U_2 of ω_0 such that u is real analytic in U_2. In [2], the following criterion for analytic hypoellipticity at a point is proved:

Theorem B. *Let* $\mathscr{L}$ *satisfy* (i)–(iv) *of* § 1, *and suppose that* $Z_1,\ldots,Z_N$ *form a basis of* $\mathscr{L}$. *Let the* φ_{lk} *be as in the representation of the* Z_j *in* (3.2). *Suppose that:*

(3.6) Any *(nontrivial) real linear combination* $\sum a_{lk}\operatorname{Re}(\varphi_{lk})$ *satisfies the sector property.*

Then $\mathscr{L}$ *is analytic-hypoelliptic at* ω_0.

We shall sometimes write (3.6) as "$\mathscr{L}$ satisfies the sector property".

§ 4. Reduction to a CR system

The two lemmas proved in this section accomplish the first step in the proof of sufficiency in Theorem 2: reduction to the case where $\mathscr{L}\cap\bar{\mathscr{L}}=(0)$.

(4.1) **Lemma.** *Let* $\mathfrak{g}$, $\mathbb{L}$ *be as in Section* 1; *suppose that* $\mathbb{L}$ *satisfies* (H1) *and that* $\mathbb{L}$ *has a basis* $L_1,\ldots,L_N$ *such that*

(4.2) $$\frac{1}{i}\sum_{j=1}^{N}[L_j,\bar{L}_j]=\sum_{j,k=1}^{N}\operatorname{Re}\alpha_{jk}[L_j,L_k]$$

for appropriate complex constants α_{jk}. *Then there exist a Lie algebra* $\mathfrak{g}'=\mathfrak{g}\oplus\mathbb{R}^K/(\mathfrak{g}'=\mathfrak{g}'_1\oplus\mathfrak{g}_2\oplus\ldots\oplus\mathfrak{g}_r$, with $\mathfrak{g}'_1=\mathfrak{g}_1\oplus\mathbb{R}^K)$ *and complex vectors* $L'_1,\ldots,L'_N\in\mathfrak{g}'\otimes\mathbb{C}$, *such that if* $\mathbb{L}'$ *is the span of* $L'_1,\ldots,L'_n$, *and if* $\pi\colon G'\to G$ *is the natural projection, then*

(a) $\mathbb{L}'\cap\bar{\mathbb{L}}'=(0)$;

(b) $(L_jf)\circ\pi=(L'_j(f\circ\pi))$, *all* $f\in C^\infty(G)$ (i.e., L'_j *restricted to* Ω *is* L);

(c) $\mathbb{L}'$ *satisfies* (H1) *and* (4.2) *on* G'.

Proof. Let $\mathbb{L}\cap\bar{\mathbb{L}}=\mathbb{M}$, let $M_1^1,\ldots,M_K^1$ be a basis of $\mathbb{M}$ consisting of real vectors (i.e., elements of $\mathfrak{g}_1$), and let $L_1^1,\ldots,L_J^1$ be a supplementary basis to $\mathbb{M}$ in $\mathbb{L}$. Then the vectors $\operatorname{Re}L_j^1$, $\operatorname{Im}L_k^1$ are linearly independent. Let G have the natural homogeneity determined by $\mathfrak{g}$. Then we may give G homogeneous coordinates (x_{jk}), where x_{jk} is of homogeneous degree j, such that

$$M_1^l=\frac{\partial}{\partial x_{1,2J+l}}+\sum_{j>1}a_{jk}^l(x)\frac{\partial}{\partial x_{jk}}\quad(1\leq j\leq K),$$

and such that no term involving any $\dfrac{\partial}{\partial x_{1,2J+s}}$ appears in the expressions for the $\operatorname{Re} L_j^1$, $\operatorname{Im} L_k^1$. Now let $G' = G \times \mathbb{R}^K$, and define

$$L_1'^j = L_1^j \quad (1 \leqq j \leqq J), \qquad M_1'^j = M_1^j + i\frac{\partial}{\partial t_j} \quad (1 \leqq j \leqq k).$$

It is easy to verify that (a), (b), (c) hold for $\mathbb{L}'$.

(4.3) **Lemma.** *Let* $\mathfrak{g}, \mathbb{L}$ *satisfy the hypotheses of Lemma* 4.1, *and let* $\mathcal{L}$ *be the Lie subalgebra of* $\mathfrak{g}\otimes\mathbb{C}$ *generated by* $\mathbb{L}$. *Then there exist complex vector fields* $\tilde{L}_j$ $(1 \leqq j \leqq N)$ *on* $\Omega' = G \times \mathbb{R}^R$ *(for appropriate* $\mathbb{R}$*) such that if* $\tilde{\mathcal{L}}$ *is the Lie algebra of vector fields generated by* $\tilde{L}_1, \ldots, \tilde{L}_N$ *and if* $\pi\colon \Omega \to G$ *is the natural projection, then*

(a) $(L_j f)\circ\pi = \tilde{L}_j(f\circ\pi)$;

(b) $\tilde{\mathcal{L}} \cap \overline{\tilde{\mathcal{L}}} = (0)$;

(c) *there are dilations (determined by the exponential coordinates given by the real and imaginary parts of* $\tilde{L}_1, \ldots, \tilde{L}_n$*) such that*

$$\sum_{j=1}^{n} [\tilde{L}_j, \overline{\tilde{L}}_j] = \mathcal{O}(-1).$$

Proof. From Lemma 4.1, we may assume that $\mathbb{L} \cap \bar{\mathbb{L}} = (0)$, and we do so. Let $\mathcal{M} = \mathcal{L} \cap \bar{\mathcal{L}}$; let $M_1, \ldots, M_R$ be a basis of $\mathcal{M}$ consisting of real vector fields, with M_i homogeneous of degree $m_i \leqq r$. Let $\{S_{jk}\}$ be a set of homogeneous elements of $\mathfrak{g}$ extending $\{\operatorname{Re} L_j, \operatorname{Im} L_k, M_1, \ldots, M_R\}$ to a basis of $\mathfrak{g}$. In the corresponding exponential coordinates, we may write

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{l,k} a_{lk}^j(z,\bar{z})\frac{\partial}{\partial s_{lk}} + \sum_k b_k^j(z,\bar{z})\frac{\partial}{\partial w_k}, \tag{4.4}$$

where s_{lk} corresponds to S_{lk}, w_k corresponds to M_k, and z_i corresponds to L_j. Hence we may find c_{lk}^j, d_k^j such that

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{l,k} c_{lk}^j(z,\bar{z})\frac{\partial}{\partial s_{lk}} + \sum_k d_k^j(z,\bar{z}) M_k. \tag{4.5}$$

For any $Z \in \mathfrak{g}\otimes\mathbb{C}$, let $\tilde{Z}$ be the vector field on $\Omega' = G \times \mathbb{R}^R$ obtained by replacing each $\dfrac{\partial}{\partial w_k}$ with $\dfrac{\partial}{\partial w_k} + i\dfrac{\partial}{\partial t_k}$; let $\tilde{\mathcal{L}}$ be the lifting of $\mathcal{L}$. Then $\tilde{\mathcal{L}}$ is again an algebra, and (a) and (b) evidently hold. We need only verify (c). We have, from (4.5),

$$\tilde{L}_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{l,k} c_{lk}^j(z,\bar{z})\frac{\partial}{\partial s_{lk}} + \sum_k d_k^j(z,\bar{z})\tilde{M}_k.$$

Hence, since $M_k = \bar{M}_k$,

$$\overline{\tilde{L}}_j = \tilde{L}_j + \sum_k \overline{d_k^j(z,\bar{z})}(\overline{\tilde{M}}_k - \tilde{M}_k), \tag{4.6}$$

so that

$$(4.7)\qquad \frac{1}{i}\sum_j[\tilde{L}_j,\bar{\tilde{L}}_j]=\frac{1}{i}\Big(\sum_j[\tilde{L}_j,\bar{L}_j]+\sum_{j,k}(\tilde{L}_j\bar{d}_k^j)(\bar{\tilde{M}}_k-\tilde{M}_k)\Big)$$
$$+\Big(\sum_{j,k}\bar{d}_k^j[\tilde{L}_j,\bar{\tilde{M}}_k-\tilde{M}_k]\Big).$$

We determine the homogeneity by giving degree -1 to the $\operatorname{Re}\tilde{L}_j$, $\operatorname{Im}\tilde{L}_j$, $\operatorname{Re}\tilde{M}_k$, $\operatorname{Im}\tilde{M}_k$, and to the real and imaginary parts of the commutators

$$[\tilde{L}_{j_1},[\tilde{L}_{j_2},\ldots,[\tilde{L}_{j_{k-1}},\tilde{L}_{j_k}]\ldots].$$

We need to show that each of the three sums on the right of (4.7) has degree $\geqq -1$. Since each $\bar{\tilde{M}}-\tilde{M}_k$ has degree -1, the middle sum has degree $\geqq -1$. Furthermore, $[\tilde{L}_j,\bar{\tilde{M}}_k-\tilde{M}]$ has degree $\geqq -2$, and $\bar{d}_k^j(0)=0$ $\left(\text{since at } 0,\ L_j=\frac{\partial}{\partial \bar{z}_j}\right)$; thus each $d_k^j(z,\bar{z})$ has degree $\geqq 1$, and the last sum has degree $\geqq -1$.

That leaves the first sum,

$$(4.8)\qquad \frac{1}{i}\sum_{j=1}^{N}[\tilde{L}_j,\bar{L}_j]=\frac{1}{i}\sum_j[\widetilde{L_j,\bar{L}_j}]$$
$$=\tfrac{1}{2}\Big(\sum_{j,k}\alpha_{jk}[\tilde{L}_j,\bar{\tilde{L}}_k]+\sum_{j,k}\alpha_{jk}[\tilde{L}_j,\bar{\tilde{L}}_k]\Big)$$

from (4.2). The first sum on the right side of (4.8) is of degree -1, by construction. As for the second sum, (4.6) and the above analysis give

$$[\tilde{L}_j,\bar{\tilde{L}}_k]=[\bar{\tilde{L}}_j,\bar{\tilde{L}}_k]+\mathcal{O}(-1)=[\overline{\tilde{L}_j,\tilde{L}_k}]+\tilde{\mathcal{O}}(-1),$$

where $\tilde{\mathcal{O}}(-1)$ is again a term of degree -1. Thus the second sum is also of degree $\geqq -1$, and the lemma follows.

Remarks. (a) If $\tilde{\mathcal{L}}$ is analytic-hypoelliptic at 0 in Ω', then $\mathcal{L}$ (hence $\mathbb{L}$) is analytic-hypoelliptic at 0 in G, from (a). The left invariance of $\mathbb{L}$ lets us conclude that $\mathbb{L}$ is analytic hypoelliptic on G.

(b) It is easy to see that the $A(\Omega')$-module generated by $\mathcal{L}$ has properties (i)–(iv) of § 1 (with $\omega_0=0$).

§ 5. Analytic hypoellipticity for vector fields with a CR structure

The results of the previous section show that Theorem 2 is a consequence of the following generalization of Theorem 3:

Theorem 3'. *Let $\mathcal{L}$ be an* $A(\Omega)$-module *of real analytic complex vector fields on an open set $\Omega\subset\mathbb{R}^M$ which satisfies properties* (i)–(iv) *of Section* 1. *Suppose further that there is a basis $L_1, L_2, \ldots, L_n$ of $\mathcal{L}$ such that*

(5.1) $$\sum_{j=1}^{N} [L_j, \bar{L}_j] = \mathcal{O}(-1),$$

where $\mathcal{O}(-1)$ *is of weight* $\geqq -1$ *in the homogeneity determined by the* L_l. *Then* $\mathcal{L}$ *is analytic-hypoelliptic at* ω_0.

Proof. We apply Theorem B (Section 3). Denote by L_j^1 the homogeneous part of L_j. We have (see (3.2))

(5.2) $$L_j^1 = \frac{\partial}{\partial \bar{z}_j} + i \sum \frac{\partial \varphi_{lk(z,\bar{z})}}{\partial z_i} \frac{\partial}{\partial s_{lk}}.$$

It follows immediately from (5.1) and homogeneity arguments that

(5.3) $$\sum_{j=1}^{n} [L_j^1, \bar{L}_j^1] = 0.$$

Since

(5.4) $$[L_j, \bar{L}_j] = 2i \sum \frac{\partial^2 \varphi_{lk}}{\partial z_j \partial \bar{z}_j} \frac{\partial}{\partial s_{lk}}.$$

We conclude from (5.2) and (5.3) that each φ_{lk} must be harmonic. Moreover, no nontrivial linear combination of the φ_{lk} (for fixed l) is pluriharmonic. Indeed, suppose otherwise. Then by a linear change of coordinates we may assume some φ_{l_0,k_0} is pluriharmonic. Then it is easy to see that condition (iii) of Section 1 is violated in the direction dual to $\dfrac{\partial}{\partial s_{l_0k_0}}$.

In view of Theorem B, Theorem 3′ is proved once we show the following:

(5.5) **Proposition.** *Let* $p(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n)$ *be a homogeneous realvalued polynomial which is harmonic but not pluriharmonic. Then* p *has the sector property.*

Proof. We use the criterion of Proposition A (Section 3). Let p be homogeneous of degree m, and define $\tilde{p} = \tilde{p}_m$ as in (3.4). For each $\omega \in S^{2n-1}$ (the unit sphere in $\mathbb{C}^n \cong \mathbb{R}^{2n}$), let f_ω: $\mathbb{C} \to \mathbb{C}^n$ be the complex linear map taking 1 to ω. By the mean value property for harmonic functions,

$$\int_{S^{2n-1}} \tilde{p}(f_\omega, 1) d\omega = 2p(0) = 0.$$

This shows that if $\tilde{p}$ is ever positive, it is also negative, and conversely. Thus either p has the sector property (by Proposition A) or $\tilde{p}(f_\omega, 1) = 0$ for all ω. Since $\tilde{p}(f_\omega, \theta) = \tilde{p}(f_{\omega'}, 1)$ with $\omega' = \omega e^{i\theta}$,

(5.6) $$\tilde{p}(f_\omega, \theta) = 0 \quad \text{for all } \omega, \theta.$$

Now let q be a homogeneous polynomial of degree m on $\mathbb{C} \cong \mathbb{R}^2$ such that $q(z, \bar{z}) + q(ze^{i\pi/m}, e^{-i\pi/m}) = 0$ for all $z \in S^1$. On S^1, we may write $q = q(\cos\theta, \sin\theta)$. Then q is periodic with period $2\pi/m$, so that q is a function of $\cos m\theta$ and $\sin m\theta$ of degree m in $\cos\theta$, $\sin\theta$. Thus

$$q(\cos\theta, \sin\theta) = \alpha\cos m\theta - \beta\sin m\theta = \operatorname{Re}(\gamma e^{im\theta}), \qquad \gamma = \alpha + i\beta,$$

or $q(z, \bar{z}) = \operatorname{Re}\gamma z^m$.

We return to the function p satisfying (5.6). Let $\omega = (\omega_1, \ldots, \omega_m)$. Then the above analysis shows that

$$(5.7) \qquad p(\omega_1 z, \omega_2 z, \ldots, \omega_n z, \bar{\omega}_1 \bar{z}, \ldots, \bar{\omega}_n \bar{z}) = \tfrac{1}{2}(\gamma(\omega) z^m + \gamma(\bar{\omega}) \bar{z}^m)$$

where γ is a function. On the other hand, if

$$p(z_1, \ldots, \bar{z}_n, \bar{z}_1, \ldots, \bar{z}_n) = \sum_{|\alpha|+|\beta|=m} C_{\alpha\beta} z^\alpha \bar{z}^\beta,$$

then

$$(5.8) \qquad p(\omega_1 z, \ldots, \omega_n z, \bar{\omega}_1 \bar{z}, \ldots, \bar{\omega}_n \bar{z}) = \sum_{|\alpha|+|\beta|=m} C_{\alpha,\beta} \omega^\alpha \bar{\omega}^\beta z^{|\alpha|} \bar{z}^{|\beta|}.$$

Comparing (5.7) with (5.8), we see that $C_{\alpha,\beta} = 0$ unless $|\alpha| = 0$ or $|\beta| = 0$. Hence

$$\gamma(\omega_1, \ldots, \omega_n, \bar{\omega}_1, \ldots, \bar{\omega}_n) = \sum_{\alpha,\beta} C_{\alpha,0}\omega^\alpha + C_{0,\beta}\bar{\omega}^\beta$$

and p is pluriharmonic.

This completes the proof of Proposition 5.6 and of Theorem 3′.

Proof of Theorem 3. The theorem follows once we note that the elements of $\mathscr{L}$ and $\bar{\mathscr{L}}$ have weight $\geqq -1$.

Proof of Theorem 2. As shown in Section 4, we can lift $\mathscr{L}$ (on G) to $\tilde{\mathscr{L}}$ on Ω', where $\tilde{\mathscr{L}} \cap \bar{\tilde{\mathscr{L}}} = (0)$. Furthermore, if $L_1, \ldots, L_N$ is a basis of $\mathbb{L}$, then the lifted vector fields $\tilde{L}_1, \ldots, \tilde{L}_N$ satisfy

$$(5.9) \qquad \frac{1}{i}\sum_{j=1}^{N} [\tilde{L}_j, \bar{\tilde{L}}_j] = \mathcal{O}(-1).$$

Now complete $L_1, \ldots, L_N$ to a basis $L_1, \ldots, L_n$ of $\mathscr{L}$ with $L_j \in \mathfrak{g}_2 \otimes \mathbb{C}$ for $N < j \leqq n$, and lift to $\tilde{L}_1, \ldots, \tilde{L}_n$. Because the condition $[\mathfrak{g}^2, \mathfrak{g}^2] = 0$ lifts we have

$$[\tilde{L}_j, \bar{\tilde{L}}_j] = 0 \qquad \text{if } N < j \leqq n;$$

hence

$$(5.10) \qquad \frac{1}{i}\sum_{j=1}^{N} [\tilde{L}_j, \bar{\tilde{L}}_j] = \mathcal{O}(-1).$$

Now Theorem 3′ implies that $\tilde{\mathscr{L}}$ is analytic hypoelliptic at 0. As remarked in Section 4, this implies Theorem 2.

(5.11) *Example.* Let M be a hypersurface in $\mathbb{C}^{N+1}$ defined by

$$\operatorname{Im} z_{N+1} = \Phi(z, \bar{z}), \qquad z \in \mathbb{C}^N.$$

Suppose that Φ is a homogeneous polynomial of degree r which is harmonic but not pluriharmonic. Then Proposition 5.5 implies that the system of tangential Cauchy-Riemann equations for M is analytic-hypoelliptic at 0. Hence (by [31]) any CR function on M extends to a holomorphic function on $\mathbb{C}^N$ in a neighborhood of the origin.

§6. Proof of Proposition (1.1)

Choose bases $\{L_1, \ldots, L_n\}$ and $\{Y_1, \ldots, Y_{m_2}\}$ for $\mathbb{L}$ and $\mathfrak{g}_2$ respectively, such that $\{Y_{m+1}, \ldots, Y_{m_2}\}$ forms a basis for $\mathrm{Re}[\mathbb{L},\mathbb{L}]+\mathrm{Im}[\mathbb{L},\mathbb{L}]$. Write

$$\frac{1}{i}[L_j, \bar{L}_k] = \sum_{l=1}^{m_2} a_{jk}^l Y_l \tag{6.1}$$

and let A_l be the $n\times n$ matrix (a_{jk}^l), $1\leqq j,k\leqq n$. The A_l are Hermitian, and the matrix for the Levi form $\langle\, ,\rangle_\lambda$ with respect to the basis $\{L_1, L_2, \ldots, L_n\}$ is $\sum_{l=1}^{m} \lambda(Y_l)A_l$ (we assume that $\lambda\perp(\mathrm{Re}[\mathbb{L},\mathbb{L}]+\mathrm{Im}[\mathbb{L},\mathbb{L}])$. A change of basis for $\mathbb{L}$ changes A_l to $A_l' = TA_lT^*$, where T is the matrix in $GL(n,\mathbb{C})$ which implements the change of basis. Since Proposition (1.1) evidently holds if the $A_l (1\leqq l\leqq m)$ have trace zero, the following suffices to complete the proof.

(6.2) **Lemma.** *Let $\mathscr{A}$ be a (real) linear subspace of the vector space H_n of $n\times n$ Hermitian matrices. Suppose that every non zero $A\in\mathscr{A}$ has a negative eigenvalue. Then there exists $T\in GL(n,\mathbb{C})$ such that* $\mathrm{Tr}(TAT^*)=0$ *for all $A\in\mathscr{A}$.*

Proof. Let $\mathscr{P}_n$ be the set of positive definite matrices in H_n, and denote the standard inner product in H_n by $\langle T_1, T_2\rangle = \mathrm{Tr}(T_2T_1)$. Choose a basis $\{A_1, A_2, \ldots, A_m\}$ for $\mathscr{A}$ and define a linear map φ: $H_n\to\mathbb{R}^m$ by

$$\varphi(B) = (\langle A_1, B\rangle, \ldots, \langle A_m, B\rangle). \tag{6.3}$$

Set $S=\varphi(\mathscr{P}_n)$. If the conclusion of the lemma is false, then for every $T\in GL(n,\mathbb{C})$ there exists j such that $0\neq\mathrm{Tr}(TA_jT^*) = \langle A_j, T^*T\rangle$; this implies easily that $0\notin S$. But $\mathscr{P}_n$ is convex and φ is linear. Hence S is convex. Therefore there is a hyperplane V such that S lies in one of the two (closed) half spaces determined by V (see e.g. [6, Theorem 7]. Equivalently, there exists a nonzero $v=(v_1, v_2, \ldots, v_m)\in\mathbb{R}^m$ such that

$$\sum v_j u_j \geqq 0 \quad \text{for every } u=(u_1, u_2, \ldots, u_m)\in S. \tag{6.4}$$

Let $A=\sum_{j=1}^{m} v_j A_j$. Then (6.4) implies that

$$\langle A, T\rangle \geqq 0 \quad \text{for all } T\in\mathscr{P}_n.$$

But the hypothesis implies that there is a unitary matrix U such that

$$U^*AU = \mathrm{Diag}(\varepsilon_1, \ldots, \varepsilon_n) \quad \text{with } \varepsilon_1 < 0.$$

Let $T_0 = \mathrm{Diag}(M, 1, 2, \ldots, 1)$ with M sufficiently large ($n\|A\||\varepsilon_1^{-1}|$ will suffice). Then

$$\langle A, UT_0U^* \rangle = \langle U^*AU, T_0 \rangle < 0,$$

and $UT_0U^* \in \mathscr{P}_n$. This gives a contradiction, and the proof of the lemma is complete.

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