SMOOTHNESS OF SOLUTIONS OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS CONSTRUCTED FROM VECTOR FIELDS

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ABSTRACT. Sufficient conditions are given for hypoellipticity of operators of the form

\[ \sum_{k=1}^{d} a_{i_1 i_2 \ldots i_k} (x) x_{i_1} x_{i_2} \ldots x_{i_k} \]

where the \( x_{i_s} \) are smooth, real vector fields and each \( a_{i_1 i_2 \ldots i_k} \) is a smooth, complex valued function. Hypoellipticity of \( L \) is related to that of a corresponding left invariant operator on a nilpotent group.

1. INTRODUCTION AND MAIN RESULT. This is an announcement of new results on hypoellipticity of certain partial differential operators; details will be published elsewhere [9]. Suppose \( X_1, X_2, \ldots, X_p \) are real, smooth vector fields on a manifold \( M \) satisfying the following two conditions.

(1.1) The \( \{X_k\} \) together with their commutators up to some fixed length \( r \) span the tangent space at each point.

(1.2) For each \( j \leq r \) the dimension of the space spanned by the commutators of \( \{X_k\} \) of length \( < j \) is constant. The class of operators to be considered here consists of those of the form

\[ L = \sum_{s \leq d} a_{i_1 i_2 \ldots i_s} (x) x_{i_1} x_{i_2} \ldots x_{i_s} \]

where the coefficients \( a_{i_1 i_2 \ldots i_s} \) are complex valued smooth functions.

The condition (1.1) was introduced by Hörmander [6], who proved that if \( \{X_k\} \) satisfies (1.1), then \( L = \sum_{j=1}^{n-1} x_j^2 + x_n \) is hypoelliptic. The additional condition (1.2) was given by Métivier [7], who showed that (1.1) and (1.2) guarantee an "approximation" of \( \{X_k\} \) by a set of vector fields generating a nilpotent Lie algebra. We review this construction.


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Let \( x_0 \) be fixed and choose vector fields \( X_{jk} \) \( j < r \) such that each \( X_{jk} \) is a commutator of length \( j \) of the \( \{ X_k \} \), and such that for each \( x \) near \( x_0 \) and each \( j_0 < r \) \( \{ X_{jk}(x) : j < j_0 \} \) is a basis for the subspace spanned by all commutators of length \( < j \), and \( X_{1k} = X_k \). Now introduce local coordinates for \( y \) varying near \( x \) as follows,

\[
\theta(x,y) = u = (u_{jk}) \text{ if } \exp_x \int u_{jk} X_{jk} = y,
\]

where \( \exp \) denotes the exponential mapping.

On \( \mathbb{R}^n \) with coordinates \( u = (u_{jk}) \) we introduce the family of dilations \( \delta_x, x, r > 0 \), defined by \( \delta_x(u_{jk}) = (r^ju_{jk}) \). A function \( f(u) \) is then homogeneous of degree \( s \) if \( f(\delta_x u) = r^sf(u) \). A vector field is homogeneous of degree \( t \) if it is a linear combination of terms of the form \( f(u) \frac{\partial}{\partial u_{jk}} \) with \( f \) homogeneous of degree \( s \) and \( t = j-s \); it is of local degree \( < t \) if its Taylor expansion is a sum of terms homogeneous of degrees \( < t \).

Now for any \( k \), let

\[
X_k, x = (d\theta)^s_x (X_k),
\]

the image of \( X_k \) under the differential of the mapping \( y \to \theta(x,y) \) at \( y = x \). Metivier has shown that

\[
(1.3) \quad X_k, x = \hat{X}_{k,x} + R_{k,x},
\]

with \( \hat{X}_{k,x} \) homogeneous of degree \( 1 \) and \( R_{k,x} \) of local degree \( < 0 \).

Furthermore, \( \hat{X}_{k,x} \) generates a nilpotent Lie algebra \( \mathcal{U}^X_k \) of step \( r \). In face \( \mathcal{U}^X_k \) is stratified, i.e.

\[
\mathcal{U}^X_k = \mathcal{U}^1_k + \mathcal{U}^2_k + \ldots + \mathcal{U}^r_k,
\]

a linear direct sum, with \( \mathcal{U}^1_k, \mathcal{U}^2_k \subset \mathcal{U}^j_k \). We may also assume \( \hat{X}_{k,x} \) is a basis of \( \mathcal{U}^1_k \). Note that \( \mathcal{U}^X_k \) carries a family of dilations \( \delta_x \) which are automorphisms: \( \delta_x(Y) = r^jY \) for \( Y \in \mathcal{U}^j_k \). Homogeneity with respect to these dilations agrees with the previous definition.

**THEOREM 1.** Suppose \( \{ X_k \} \) are real smooth vector fields on a manifold \( M \) satisfying (1.1) and (1.2) near \( x_0 \) and

\[
L = \sum_{k \leq d} a_{i_1i_2\ldots i_k} (x) X_{i_1} X_{i_2} \ldots X_{i_k}. \quad \text{Then } L \text{ is hypoelliptic in a neighborhood of } x_0 \text{ if the homogeneous left invariant operator}
\]

\[
L_{x_0} = \sum_{k \leq d} a_{i_1i_2\ldots i_k}(x_0) \hat{X}_{i_1,x_0} \hat{X}_{i_2,x_0} \ldots \hat{X}_{i_k,x_0}
\]

is hypoelliptic.

For \( L_{x_0} \) the hypoellipticity criterion of Helffer–Nourrigat [4] may be applied: If \( D \) is a homogeneous left invariant differential operator on a nilpotent Lie group \( G \) then \( D \) is hypoelliptic if and only if for each irreducible unitary representation \( \pi \) of \( G \), \( \pi(D) \) is injective on \( S_\pi \), the space of \( C^\infty \) vectors for \( \pi \). This criterion was originally conjectured by Rockland [8].
Theorem 1 was conjectured by Helffer-Nourrigat [5], who showed that the given condition is also necessary in order that $L$ satisfy the following estimates in a neighborhood $U$ of $x_0$:

$$\sup_{s<d} \| X_1 X_2 \cdots X_s L^{-2}(U) u \|_{L^2(U)} \leq C(\| u \|_{L^2(U)} + \| u \|_{L^2(U)}).$$

2. PARAMETRICES. Theorem 1 is proved by constructing a parametrix for $L$ following the methods of Polland-Stein [2] and Rothschild-Stein [10]. Let $H^s$ denote the $L^2$ Sobolev spaces, $s \in \mathbb{R}$. Theorem 1 is a consequence of the following.

**THEOREM 2.** Suppose $L$ satisfies the hypotheses of Theorem 1. Then there exists $\varphi \in C_0^\infty(M)$ such that $\varphi \equiv 1$ on a neighborhood of $x_0$, and operators $K, S$, with $K$ bounded from $H^s$ to $H^{s+d/d}$ and $S$ given by a smooth kernel, such that

$$KLu = \varphi u + Su$$

for any compactly supported distribution $u$.

For the construction of $K$ we need the following abstract existence theorem for fundamental solutions of differential operators on nilpotent Lie groups (see [1]). Let $Q = \frac{k}{2} \dim \psi_i$ if $\psi = \psi_1 + \psi_2 + \cdots + \psi_r$. If $D$ is a hypoelliptic self-adjoint left invariant differential operator homogeneous of degree $d < Q$ on $G$, the simply connected group of $\psi$, then $D$ has a left and right inverse. More precisely, there exists $k \in C^\infty(G(0))$, homogeneous of degree $-Q + d$ such that

$$D(fk) = Df k = f$$

for all compactly supported distributions $f$.

We may reduce to the case where $F$ is self adjoint and $d < Q$ by replacing $L$ by $L^* + \sum_{j=1}^{m} \frac{2d}{\delta_j} (\frac{1}{\delta_j})$ (see [5] for details). Now suppose, in addition, that $F$ is hypoelliptic for all $x$ sufficiently close to $x_0$. Then by the above $L$ has a homogeneous fundamental solution $k_x$. Hence, as in [10, Theorem 10], a candidate for a first approximation for $K$ is

$$K_1 : f \mapsto K_1 f(x) = \int \varphi_1(x) k_x(\Theta(y,x)) \varphi_2(y) f(y) dy,$$

where $\varphi_1, \varphi_2 \in C_0^\infty$, $\varphi_2 \equiv 1$ on $\text{supp} \varphi_1$.

3. EXISTENCE AND SMOOTHNESS OF $k_x(u)$. There are two main steps in the proof of Theorem 2.

**STEP 1.** Show that $L_x$ is hypoelliptic for $x$ close to $x_0$.

**STEP 2.** Show that $(x,u) \mapsto k_x(u)$ is smooth for $u \neq 0$, $x$ close to $x_0$. 
We shall indicate briefly the construction of $k_x$. Let $K_x$ be the
operator defined initially on $C^0_0(G_x)$, where $G_x$ is the simply connected
group corresponding to $\mathcal{Y}_x$, by

$$
(3.1) \quad k_x f = \lim_{n \to \infty} \sum_{j=0}^{n} (-1)^j (f \ast k_x^j)
$$

where

$$
k_x^0 = k_{x_0}, \quad \text{and} \quad k_x^j = (L_x - l_{x_0}) k_x^0 \ast x_0 \ast (L_x - l_{x_0}) k_x^0 \ast x \ast \cdots \ast (L_x - l_{x_0}) k_x^0 \ast x_0 \ast k_x^0.
$$

Here $\ast_x$ denotes convolution over the varying group $G_x$; $K_x f$ is defined
only if the right hand side of (3.1) converges in $L^p$ for some $p$,
$1 < p < \infty$. Then $K_x f$ is the distribution $K_x f = f \ast k_x^\infty$, if defined.

One must show first that $K_x$ is defined if $f \in C^0_0$ and $|x - x_0|$ is
sufficiently small. Next, it must be shown that for fixed $x$,
$u \mapsto k_x(u)$ is smooth for $u \neq 0$ and on any compact subset the deriva-
tives are uniformly bounded in $x$. Finally, the analogous result for $u$
fixed, $u \neq 0$, must be proved for the mapping $x \mapsto k_x(u)$. Although more
complicated, most of this proof is in the spirit of that of [10, Theorem
3].

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