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SMOOTHNESS OF SOLUTIONS OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS CONSTRUCTED FROM VECTOR FIELDS

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ABSTRACT. Sufficient conditions are given for hypoellipticity of operators of the form

$$\sum_{k\le d} a_{i_1 i_2 \ldots i_k}(x) X_{i_1} X_{i_2} \ldots X_{i_k}$$

where the X_{i_s} are smooth, real vector fields and each $a_{i_1 i_2 \ldots i_k}$ is a smooth, complex valued function. Hypoellipticity of L is related to that of a corresponding left invariant operator on a nilpotent group.

1. INTRODUCTION AND MAIN RESULT. This is an announcement of new results on hypoellipticity of certain partial differential operators; details will be published elsewhere [9]. Suppose $X_1, X_2, \ldots, X_p$ are real, smooth vector fields on a manifold M satisfying the following two conditions.

(1.1) The $\{X_k\}$ together with their commutators up to some fixed length r span the tangent space at each point.

(1.2) For each $j \le r$ the dimension of the space spanned by the commutators of $\{X_k\}$ of length $\le j$ is constant. The class of operators to be considered here consists of those of the form

$$L = \sum_{s\le d} a_{i_1} a_{i_2 \ldots i_s}(x) X_{i_1} X_{i_2} \ldots X_{i_s}$$

where the coefficients $a_{i_1 i_2 \ldots i_s}$ are complex valued smooth functions.

The condition (1.1) was introduced by Hörmander [6], who proved that if $\{X_k\}$ satisfies (1.1), then $L = \sum_{j=1}^{n-1} X_j^2 + X_n$ is hypoelliptic. The additional condition (1.2) was given by Metivier [7], who showed that (1.1) and (1.2) guarantee an "approximation" of $\{X_k\}$ by a set of vector fields generating a nilpotent Lie algebra. We review this construction.

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Let x_0 be fixed and choose vector fields X_{jk} $j \le r$ such that each X_{jk} is a commutator of length j of the $\{X_k\}$, and such that for each x near x_0 and each $j_0 \le r$ $\{X_{jk}(x) : j \le j_0\}$ is a basis for the subspace spanned by all commutators of length $\le j$, and $X_{1k} = X_k$. Now introduce local coordinates for y varying near x as follows,

$$\Theta(x,y) = u = (u_{jk}) \quad \text{if} \quad \exp_x \sum u_{jk}X_{jk} = y,$$

where exp denotes the exponential mapping.

On $\mathbb{R}^n$ with coordinates $u = (u_{jk})$ we introduce the family of dilations δ_r, $r > 0$, defined by $\delta_r(u_{jk}) = (r^j u_{jk})$. A function $f(u)$ is then homogeneous of degree s if $f(\delta_r u) = r^s f(u)$. A vector field is homogeneous of degree t if it is a linear combination of terms of the form $f(u)\frac{\partial}{\partial u_{jk}}$ with f homogeneous of degree s and $t = j-s$; it is of local degree $\le t$ if its Taylor expansion is a sum of terms homogeneous of degrees $\le t$.

Now for any k, let

$$X_{k,x} = (d\Theta)^*_x(X_k),$$

the image of X_k under the differential of the mapping $y \longrightarrow \Theta(x,y)$ at $y = x$. Metivier has shown that

$$X_{k,x} = \hat{X}_{k,x} + R_{k,x}, \tag{1.3}$$

with $\hat{X}_{k,x}$ homogeneous of degree 1 and $R_{k,x}$ of local degree ≤ 0. Furthermore, $\hat{X}_{k,x}$ generates a nilpotent Lie algebra $\mathcal{G}_x$ of step r. In face $\mathcal{G}_x$ is stratified, i.e.

$$\mathcal{G}_x = \mathcal{G}^1_x + \mathcal{G}^2_x + \dots + \mathcal{G}^r_x,$$

a linear direct sum, with $[\mathcal{G}^i_x, \mathcal{G}^j_x] \subset \mathcal{G}^{i+j}_x$. We may also assume $\hat{X}_{k,x}$ is a basis of $\mathcal{G}^1_x$. Note that $\mathcal{G}_x$ carries a family of dilations δ_r which are automorphisms: $\delta_r(Y) = r^j Y$ for $Y \in \mathcal{G}^j_x$. Homogeneity with respect to these dilations agrees with the previous definition.

THEOREM 1. Suppose $\{X_k\}$ are real smooth vector fields on a manifold M satisfying (1.1) and (1.2) near x_0 and $L = \sum_{k\le d} a_{i_1 i_2 \dots i_k}(x) X_{i_1} X_{i_2} \dots X_{i_k}$. Then L is hypoelliptic in a neighborhood of x_0 if the homogeneous left invariant operator $L_{x_0} = \sum_{k=d} a_{i_1 i_2 \dots i_k}(x_0) \hat{X}_{i_1,x_0} \hat{X}_{i_2,x_0} \dots \hat{X}_{i_k,x_0}$ is hypoelliptic.

For L_{x_0} the hypoellipticity criterion of Helffer-Nourrigat [4] may be applied: If D is a homogeneous left invariant differential operator on a nilpotent Lie group G then D is hypoelliptic if and only if for each irreducible unitary representation π of G, $\pi(D)$ is injective on S_π, the space of C^∞ vectors for π. This criterion was originally conjectured by Rockland [8].

Theorem 1 was conjectured by Helffer-Nourrigat [5], who showed that the given condition is also necessary in order that L satisfy the following estimates in a neighborhood U of x_0:

$$\sup_{s\leq d} \| X_{i_1} X_{i_2} \cdots X_{i_s} \|_{L^2(U)} \leq C(\|Lu\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

2. PARAMETRICES. Theorem 1 is proved by constructing a parametrix for L following the methods of Folland-Stein [2] and Rothschild-Stein [10]. Let H^s denote the L^2 Sobolev spaces, $s \in \mathbb{R}$. Theorem 1 is a consequence of the following.

THEOREM 2. Suppose L satisfies the hypotheses of Theorem 1. Then there exists $\varphi \in C_0^\infty(M)$ such that $\varphi \equiv 1$ on a neighborhood of x_0, and operators K, S, with K bounded from H^s to $H^{s+d/r}$ and S given by a smooth kernel, such that

$$KLu = \varphi u + Su$$

for any compactly supported distribution u.

For the construction of K we need the following abstract existence theorem for fundamental solutions of differential operators on nilpotent Lie groups (see [1]). Let $Q = \sum_{i=1}^{r} i \dim \mathfrak{g}^i$ if $\mathfrak{g} = \mathfrak{g}^1 + \mathfrak{g}^2 + \cdots + \mathfrak{g}^r$. If D is a hypoelliptic self-adjoint left invariant differential operator homogeneous of degree $d < Q$ on G, the simply connected group of $\mathfrak{g}$, then D has a left and right inverse. More precisely, there exists $k \in C^\infty(G-\{0\})$, homogeneous of degree $-Q + d$ such that

$$D(f*k) = Df*k = f$$

for all compactly supported distributions f.

We may reduce to the case where $\hat{L}_x$ is self adjoint and $d < Q$ by replacing L by $L^*L + \sum_{j=1}^{m} (\frac{\partial}{\partial t_j})^{2d}$ (see [5] for details). Now suppose, in addition, that $\hat{L}_x$ is hypoelliptic for all x sufficiently close to x_0. Then by the above L_x has a homogeneous fundamental solution k_x. Hence, as in [10, Theorem 10], a candidate for a first approximation for K is

$$K_1 : f \longrightarrow K_1 f(x) = \int \varphi_1(x) k_x(\Theta(y,x)) \varphi_2(y) f(y)\, dy,$$

where $\varphi_1, \varphi_2 \in C_0^\infty$, $\varphi_2 \equiv 1$ on $\operatorname{supp} \varphi_1$.

3. EXISTENCE AND SMOOTHNESS OF $k_x(u)$. There are two main steps in the proof of Theorem 2.

STEP 1. Show that L_x is hypoelliptic for x close to x_0.

STEP 2. Show that $(x,u) \longrightarrow k_x(u)$ is smooth for $u \neq 0$, x close to x_0.

We shall indicate briefly the construction of k_x. Let K_x be the operator defined initially on $C_0^\infty(G_x)$, where G_x is the simply connected group corresponding to $\mathfrak{g}_x$, by

$$K_x f = \lim_{n\to\infty} \sum_{j=0}^{n} (-1)^j (f * k_x^j) \tag{3.1}$$

where

$$k_x^0 = k_{x_0}, \quad \text{and}$$

$$k_x^j = (L_x - L_{x_0}) k_{x_0} *_x (L_x - L_{x_0}) k_{x_0} *_x \ldots (L_x - L_{x_0}) k_{x_0} *_x k_{x_0}.$$

Here $*_x$ denotes convolution over the varying group G_x; $K_x f$ is defined only if the right hand side of (3.1) converges in L^p for some p, $1 < p < \infty$. Then k_x is the distribution $K_x f = f * k_x$, if defined.

One must show first that K_x is defined if $f \in C_0^\infty$ and $|x - x_0|$ is sufficiently small. Next, it must be shown that for fixed x, $u \longrightarrow k_x(u)$ is smooth for $u \neq 0$ and on any compact subset the derivatives are uniformly bounded in x. Finally, the analogous result for u fixed, $u \neq 0$, must be proved for the mapping $x \longrightarrow k_x(u)$. Although more complicated, must of this proof is in the spirit of that of [10, Theorem 3].

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