Reprinted from

AMERICAN
JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

THE JOHNS HOPKINS UNIVERSITY PRESS
BALTIMORE, MARYLAND 21218
U. S. A.

SOLVABILITY OF TRANSVERSALLY ELLIPTIC DIFFERENTIAL OPERATORS ON NILPOTENT LIE GROUPS

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1. Introduction. In this paper, we study some aspects of solvability of left invariant differential operators on connected, simply connected nilpotent Lie groups. In Part I, we prove a general necessary condition for microlocal solvability and apply it to the case of a transversally elliptic operator on a 2-step nilpotent Lie group. We show in Part II that for a special class of 2-step groups, the criterion of Part I is also sufficient for local solvability; the main argument here involves the division of distributions by analytic functions. Finally, Part III is devoted primarily to a sufficient condition for the existence of a global fundamental solution; this condition is satisfied for the operators discussed in Part II.

We now describe these results in more detail. Let G be a (connected, simply connected) 2-step nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (a vector space direct sum) with $\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1, \mathfrak{g}_1]$. This grading of $\mathfrak{g}$ induces a grading of the universal enveloping algebra $\mathfrak{u}(\mathfrak{g})$, $\mathfrak{u}(\mathfrak{g}) = \oplus_{j=0}^{\infty} \mathfrak{u}_j(\mathfrak{g})$. Thus a left invariant differential operator $L \in \mathfrak{u}(\mathfrak{g})$ can be written $L = \Sigma_{j=0}^{m} L_j$, with L_j homogeneous of degree j. In the homogeneous case ($L = L_m$), criteria for local solvability of L have been given by the authors individually ([21], [3], [4]) and severally ([5]), by Levy-Bruhl ([12], [13], [14]), by Rothschild-Tartakoff ([24]), and by others; a survey of some of these results is found in [18]. In Parts I and II of this paper, we use new techniques to obtain results for non-homogeneous operators.

We say that L is *transversally elliptic* (or *elliptic in the generating directions*) if for every nontrivial 1-dimensional unitary representation σ of G, $\sigma(L_m) \neq 0$. (Note that $\sigma \equiv 0$ on $\mathfrak{g}_2$.) A differential operator P is *locally solvable* at x_0 if there is a neighborhood U of x_0 such that the equation

Manuscript received 29 December 1983.

[1]Supported by NSF Grants MCS-82-02243.
[2]Supported by NSF Grants MCS-82-03949.

American Journal of Mathematics 108 (1986), 589–614.

$Pu = h$ has a solution $u \in \mathcal{C}^\infty(U)$ for each $h \in \mathcal{C}_c^\infty(U)$. The main result of Part I is the following:

THEOREM 1.1. *Let G be a 2-step nilpotent Lie group, and let $L \in \mathfrak{u}(\mathfrak{g})$ be transversally elliptic. Suppose that there exists a nonzero $f \in \mathcal{L}^2(G)$ such that $L^\tau f = 0$. Then L is not locally solvable at 0.*

The proof runs roughly as follows: we first show that if $\mathcal{L}^2(G) \cap \operatorname{Ker} L^\tau \neq (0)$, then there is a map $\Pi: \mathcal{L}^2(G) \to \operatorname{Ker} L^\tau$ such that $\Pi = \Pi^\tau$ and Π is pseudolocal but not infinitely smoothing. Thus there exists $f \in \mathcal{L}^2(G)$ with $\Pi f \notin C^\infty$. Suppose that Πf is not $\mathcal{C}^\infty$ near 0. If one can solve $Lu = f$ near 0, then $f - Lu$ is C^∞ near 0; hence

$$\Pi(f - Lu) = \Pi f - (L^\tau \Pi)^\tau u = \Pi f$$

is $\mathcal{C}^\infty$ near 0, a contradiction. Since local solvability on G in $\mathcal{C}^\infty$ implies local solvability in $\mathcal{L}^2$ (see [22], Section 14), L is not locally solvable. (This line of reasoning was suggested by arguments in [7].) In Section 2, we phrase this argument in terms of microlocal solvability; we apply it to our situation in Section 3 and Section 4.

In Part II, we specialize to the case where G is an (H)-group; i.e., we assume that for every $\eta \in \mathfrak{g}_2^*\backslash\{0\}$, the bilinear form B_η on $\mathfrak{g}_1 \times \mathfrak{g}_1$ defined by

$$B_\eta(X, Y) = \eta([X, Y]) \tag{1.2}$$

is nondegenerate. For (H)-groups, the infinite dimensional irreducible unitary representations are parametrized by $\mathfrak{g}_2^*\backslash\{0\}$. Let π_η denote the representation corresponding to $\eta \in \mathfrak{g}_2^*\backslash\{0\}$. Our main result is

THEOREM 1.3. *If G is an (H)-group and $L \in \mathfrak{u}(\mathfrak{g})$ is transversally elliptic, then the following are equivalent:*

(i) $\operatorname{Ker} L^\tau \cap L^2(G) = (0)$;
(ii) *L is locally solvable*;
(iii) *there is no open set $U \subset \mathfrak{g}^*$ such that* $\operatorname{Ker} \pi_\eta(L^\tau) \neq 0$ *for all* $\eta \in U$.

That (ii) implies (i) follows from Part I; the equivalence of (i) and (iii) is standard. To show that (i) implies (ii), we combine an argument of Melin [16] with the notion of dividing a distribution by an analytic function, as in [24].

In Part III, we give a sufficient condition for $L \in \mathfrak{u}(\mathfrak{g})$ to have a global fundamental solution—i.e., a distribution σ with $L\sigma = \delta$ on G. The condition is that L is uniformly semiglobally solvable; i.e., that there is an integer r such that for every bounded open neighborhood U of the identity there exists a distribution σ_u of order at most r such that $L\sigma_u = \delta$ in U. We apply this result to show that the conditions of Theorem 1.3 are equivalent to

(1.4) $\quad L$ has a global fundamental solution.

Another simple consequence is that if L is homogeneous and locally solvable, then L has a global fundamental solution. The basic result of this section, incidentally, applies to all Lie groups G diffeomorphic to some $\mathbf{R}^n$.

Some examples and open questions are discussed in Section 8.

Part I. Necessary Conditions for Microlocal and Local Solvability

Let U be an open set in $\mathbf{R}^n$. We denote by $\mathcal{D}'(U)$ the set of distributions on U and by $\mathcal{E}'(U)$ the subspace of distributions with compact support in U. The wavefront set of $u \in \mathcal{D}'(U)$, written $WF\, u$, is the complement of

$$\{(x, \xi) \in T^*U \setminus \{0\}\colon \text{there exist } \phi \in C_c^\infty(\mathbf{R}^n) \text{ and constants } C_N \text{ with}$$

$$\phi(x) \neq 0,\ |(\phi u)\hat{}(\eta)| \leq C_N(1 + |\eta|)^{-N}$$

$$\text{for all } \eta \text{ in an open cone containing } \xi \text{ and for all } N\}.$$

An operator $Q\colon \mathcal{E}'(U) \to \mathcal{D}'(U)$ is *microlocal* if $(x, \xi) \notin WF\, u$ implies $(x, \xi) \notin WF(Qu)$ for all $\xi \neq 0$ and all $x \in U$; Q is *regularizing* at (x, ξ) if $(x, \xi) \notin WF(Qu)$ for all $u \in \mathcal{E}'(U)$. We write $Q \sim Q'$ at (x, ξ) if $Q' - Q$ is regularizing at (x, ξ).

Now let P be a classical pseudodifferential operator on U. We say that P is *microlocally solvable* at a point $(x_0, \xi_0) \in T^*U \setminus \{0\}$ if for every distribution f on U there is a distribution u such that $(x_0, \xi_0) \notin WF(Pu - f)$.

Our reason for studying microlocal solvability is the following:

Proposition 2.1. *Let $L \in \mathfrak{u}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of a Lie group G. If L is not microlocally solvable at a point $(x_0, \xi_0) \in T^*G \setminus \{0\}$,* *then L is not locally solvable at x_0 (and hence not locally solvable anywhere).*

Proof. L is locally solvable at x_0 if and only if L has a local fundamental solution in a neighborhood of x_0 (see, e.g., Proposition 14.1 of [22]). Hence if L is locally solvable, then one can solve $Lu = f$ in a neighborhood of x_0 for any distribution f. This shows that L is microlocally solvable everywhere if L is locally solvable.

We now give a simple necessary condition for microlocal solvability.

PROPOSITION 2.2. *Let P be a properly supported classical pseudodifferential operator on an open set U. Suppose that there is a microlocal operator Π: $\mathcal{E}'(U) \to \mathcal{D}'(U)$ such that*

$$\Pi P \sim 0 \quad \text{at} \quad (x_0, \xi_0), \qquad \Pi \not\sim 0 \quad \text{at} \quad (x_0, \xi_0).$$

Then P is not microlocally solvable at (x_0, ξ_0).

Proof. Since $\Pi \not\sim 0$ at (x_0, ξ_0), there is a distribution $f \in \mathcal{E}'(U)$ such that $(x_0, \xi_0) \in WF(\Pi f)$. If there is a distribution u with

$$(x_0, \xi_0) \notin WF(Pu - f),$$

then

$$(x_0, \xi_0) \notin WF(\Pi Pu - \Pi f) = WF(\Pi f),$$

a contradiction. Hence P is not microlocally solvable.

3. Unsolvability for Grušin type Operators. As in [8], we consider operators on $\mathbf{R}^{n_1+n_2}_{t,y}$ of the form

$$P = \sum_{|\alpha+\beta| \leq m} a_{\alpha\beta}(D_y) t^\alpha D_t^\beta, \tag{3.1}$$

with pseudodifferential operators $a_{\alpha\beta}(D_y)$ whose symbols $a_{\alpha\beta}(\eta)$ are defined in a conic neighborhood of a point $\eta_0 \in \mathbf{R}^{n_2} \backslash \{0\}$. Define

$$P(\eta) = \sum_{|\alpha+\beta| \leq m} a_{\alpha\beta}(\eta) t^\alpha D_t^\beta. \tag{3.2}$$

As in [8], we assume that

$$P(\eta) = \sum_{j=0}^{m} P_j(\eta),$$

where

$$P_{m-j}(\eta) = |\eta|^{(m-j)/2} \sum_{|\alpha+\beta| \le m-j} C_{\alpha\beta}^{m-j}(\eta)(t|\eta|^{-1/2})^{\alpha}(|\eta|^{1/2}D_t)^{\beta},$$

and $C_{\alpha\beta}^{m-j}$ is analytic and homogeneous of degree 0. We also assume that P is transversally elliptic on the characteristic variety; i.e., that

$$\sum_{|\alpha+\beta|=m} C_{\alpha\beta}^{m}(\eta)(t')^{\alpha}(\tau')^{\beta} \neq 0 \tag{3.3}$$

for $(t', \tau') \in \mathbf{R}^{2n}\backslash\{0\}$, where $t' = |\eta|^{-1/2}t$ and $\tau' = |\eta|^{1/2}\tau$.

We shall use Proposition 2.2 to prove the following:

THEOREM 3.4. *Suppose that there is an open set $U \subset \mathbf{R}_\eta^{n_2}$ such that for all $\eta \in U$.*

$$\operatorname{Ker} P(\eta)^* \cap L^2(\mathbf{R}^{n_1}) \neq (0).$$

Then P is microlocally unsolvable at $(0, 0; 0, 0, \eta_0)$ for any $\eta_0 \in U$.

For the proof, we need some facts about the operators $P(\eta)$. Let $\omega = \eta/|\eta|$, $z = |\eta|^{-1/2}$. As in [8], we may write

$$z^m P(\eta) = A(z, \omega)(t', D_{t'}) = \sum_{j=0}^{m} z^j A_{m-j}(1, \omega)(t', D_{t'}),$$

where $(z, \omega) \mapsto A(z, \omega)(t', D_{t'})$ is an analytic family of unbounded operators (in the sense of Kato-Rellich; see [10]) near any point (z, ω) with $\omega = \eta/|\eta|$ and $\eta \neq 0$. If $\operatorname{Ker} P(\eta)^* \neq 0$, then of course $\operatorname{Ker} P(\eta)P(\eta)^* \neq 0$. As in [24], we consider the operator $T(\eta)$ for small $|z|$, defined by

$$T(\eta) = \frac{1}{2\pi i}\int_\Gamma (P(\eta)P(\eta)^* - \zeta I)^{-1} d\zeta, \tag{3.5}$$

where Γ is a small circle about 0 in $\mathbf{C}$. From the analyticity of the family $\eta \mapsto T(\eta)$, it follows that if $\operatorname{Ker} P(\eta)P(\eta)^* \neq (0)$ for all η in a connected open set U, then Ker $P(\eta)P(\eta)^* \neq (0)$ for all η in the connected component of $\mathbf{R}^{n_2}\setminus\{0\}$ containing U. (See the proof of Proposition 7.1 in [24].) We shall need the following result:

PROPOSITION 3.6. *Fix η_0. Under the hypotheses of Theorem 3.4 there is an analytic family of vectors $v_{(z,\omega)}(u) \in L^2(\mathbf{R}^{n_1})$ such that*

$$P(\eta)^* v_{(z,\omega)}(|\eta|^{1/2}t) \equiv 0$$

whenever $|z|$ and $|\eta/|\eta| - \eta_0/|\eta_0||$ are small.

Proof. Note first that $T(\eta)$ is a projection onto a finite-dimensional subspace $\mathcal{H}^1_{(z,\omega)}$ of $L^2(\mathbf{R}^{n_1})$, that $T(\eta)$ commutes with $P(\omega)P^*(\omega)$, and that $T(\eta)$ varies analytically with η. As in Kato ([10], Chapter VII, Section 3, proof of Theorem 1.7), we may assume after an analytic transformation that $\mathcal{H}^1_{(z,\omega)} = \mathcal{H}^1$, independent of (z, ω). Now the proof of the proposition reduces to the following result for finite-dimensional spaces.

LEMMA 3.7. *Let $B(\zeta)$ be an analytic family of $N \times N$ matrices acting on a (complex) vector space V, with $B(\zeta)$ diagonalizable for ζ real. Suppose that* Ker $B(\zeta) \neq 0$ *for all ζ. Then for a fixed real ζ_0 there exists an analytic family of vectors $\zeta \mapsto w_\zeta$, defined for $|\zeta - \zeta_0|$ small, such that*

$$B(\zeta)w_\zeta \equiv 0 \quad \textit{for} \quad |\zeta - \zeta_0| \quad \textit{small}, \quad w_\zeta \not\equiv 0. \tag{3.8}$$

Proof. For convenience, let $\zeta_0 = 0$. Let K be the generic multiplicity of 0 as an eigenvalue of $B(\zeta)$, $|\zeta|$ small, and let $\lambda_1(\zeta), \ldots, \lambda_{N-K}(\zeta)$ be the other eigenvalues of $B(\zeta)$. Let $Q(\zeta)$ be the projection onto the space of zero eigenvectors away from exceptional points, and define $\tilde{Q}(\zeta)$ by

$$\tilde{Q}(\zeta) = \prod_{j=1}^{N-K} \lambda_j^N(\zeta)Q(\zeta). \tag{3.9}$$

We shall show that $\tilde{Q}$ extends to an analytic family of operators near 0. Assume this for the moment. Then, since $\tilde{Q} \not\equiv 0$, there is a vector $w \in V$ such that $\tilde{Q}(\zeta)w \neq 0$. The vectors $w_\zeta = \tilde{Q}(\zeta)w$ form an analytic family satisfying (3.9).

Thus we need to show that (3.9) is analytic, or that for any $w_1, w_2 \in V$, the mapping $\zeta \to (\tilde{Q}(\zeta)w_1, w_2)$ is an analytic function from an open set in $\mathbf{C}^n$ to $\mathbf{C}$. If $\zeta_1, \ldots, \zeta_{i-1}, \zeta_{i+1}, \ldots, \zeta_n$ are fixed, then $\zeta_i \mapsto (\tilde{Q}(\zeta)w_1, w_2)$ is analytic away from exceptional points (see [10], Theorem II-1.8). To prove that it is analytic everywhere, we need only show that it is uniformly bounded. For this, we use some arguments from Section II-1.5 of [10]. Let $U = \{\zeta : |\zeta| < \epsilon$ and $B(\zeta)$ has at most K zero eigenvalues$\}$. Then U is open. Let $\Gamma(\zeta)$ be a small circle in C around 0 and not enclosing any other elements of the spectrum of $B(\zeta)$. For $\zeta \in U$,

$$Q(\zeta) = \frac{1}{2\pi i} \int_{\Gamma(\zeta)} R(\zeta, \gamma) d\gamma, \tag{3.10}$$

where $R(\zeta, \gamma) = (B(\zeta) - \gamma I)^{-1}$ is the resolvent. Hence

$$\|Q(\zeta)\| \le \rho(\zeta) \max_{\gamma \in \Gamma(\zeta)} \|R(\zeta, \gamma)\|,$$

where $\rho(\zeta)$ is the radius of $\Gamma(\zeta)$. We may choose

$$\rho(\zeta) = \min_j |\lambda_j(\zeta)|/2. \tag{3.11}$$

Furthermore,

$$(B(\zeta) - \gamma I)^{-1} = (\det(B(\zeta) - \gamma I))^{-1} S(\zeta, \gamma),$$

where $S(\zeta, \gamma)$ is a polynomial in the coefficients of $B(\zeta) - \gamma I$. Since

$$\operatorname{Det}(B(\zeta) - \gamma I) = \gamma^K \prod_{j=1}^{N-K} (\lambda_j(\zeta) - \gamma)$$

and

$$|\lambda_j(\zeta) - \gamma| \ge |\lambda_j(\zeta)|/2$$

from (3.11), we have

$$|\operatorname{Det}(B(\zeta) - \gamma I)|^{-1} \le 2^N (\min_j |\lambda_j(\zeta)|)^{-N}. \tag{3.12}$$

Finally, since $\|S(\zeta, \gamma)\|$ is bounded for ζ and γ varying in a bounded set, we have

$$\max_{\gamma \in \Gamma(\zeta)} \|R(\zeta, \gamma)\| \le C (\min_j |\lambda_j(\zeta)|)^{-N},$$

and hence

$$\|\tilde{Q}(\zeta)\| \leq \prod_{j=1}^{N-K} |\lambda_j^N(\zeta)| \cdot C \min_j |\lambda_j(\zeta)|^{-N}$$

is bounded for small $|\zeta|$. This completes the proof of Lemma 3.7 and hence of Proposition 3.6.

We now prove Theorem 3.4. We do this by constructing an operator Π: $\mathcal{E}'(\mathbf{R}^{n_1-n_2}) \to \mathfrak{D}'(\mathbf{R}^{n_1+n_2})$ such that Π and Π^* are microlocal and

$$\Pi \not\sim 0, \tag{3.13}$$

near $(0, 0, 0, \eta_0)$. To construct Π, we choose a smooth cutoff function ϕ, with $\phi \equiv 0$ for η small, such that ϕ satisfies the additional properties given in [8] immediately preceding Proposition 3.15. Let

$$v_\eta'(u) = \theta(\eta) v_\eta(u).$$

Proposition 3.2 of [8] shows that v_η is a rapidly vanishing function of u for each η. Now let Π^* be the (non-classical) pseudodifferential operator defined by

$$\Pi^* f(x, t) = (2\pi)^{-n_2} \int e^{ix\cdot\eta} v_\nu'(|\eta|^{1/2} t)\left(\int e^{ix\cdot\eta} v_\eta'(|\eta|^{1/2} u) \hat{f_2}(u, \eta)\, du\right) d\eta,$$

where $\hat{f_2}$ is the partial Fourier transform of f in the second set of variables. It is proved in Proposition 3.18 of [8] that Π^* and its adjoint Π are microlocal. (In fact, Π^* may be expressed as a composition of operators of type H_i and H_i^* defined in [8].) The desired properties are then proved in [8] for the analytic wavefront set, but it is easy to modify the arguments to get corresponding results for the $\mathcal{C}^\infty$ wavefront set. Since by construction $P^*(\eta) v_\eta'(u) \equiv 0$ for η in a cone around $\eta_0 \in U$, we have

$$P^* \Pi^* \sim 0 \quad \text{near} \quad (0, 0; 0, \eta_0). \tag{3.14}$$

Now (3.13) follows from (3.14) by taking adjoints. Finally, $\Pi \not\sim 0$ near $(0, 0; 0, \eta_0)$ because $\Pi^* \not\sim 0$ there. Now Proposition 2.2 implies Theorem 3.4 immediately.

4. Local Solvability of Transversally Elliptic Operators on 2-Step Groups. Since for much of the rest of this paper we shall be dealing with 2-step nilpotent Lie groups, we begin this section by fixing some notation. Let G be a 2-step group as in Section 1, with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. We shall generally identify G with $\mathfrak{g}$ by the exponential map. A typical element of $\mathfrak{g}$ (or G) will be (x, y), $x \in \mathfrak{g}_1$ and $y \in \mathfrak{g}_2$; a typical element of $\mathfrak{g}^* = \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$ will be $\ell = (\xi, \eta)$, $\xi \in \mathfrak{g}_1^*$ and $\eta \in \mathfrak{g}_2^*$. Let $\dim \mathfrak{g} = n$, $\dim \mathfrak{g}_1 = n_1$, and $\dim \mathfrak{g}_2 = n_2$. We fix compatible Euclidean norms on $\mathfrak{g}$, $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}^*$, $\mathfrak{g}_1^*$, and $\mathfrak{g}_2^*$; these are denoted by $|\,|$.

Recall that Kirillov theory (see [11] or [20]) assigns to each $\ell \in \mathfrak{g}^*$ an irreducible unitary representation π_ℓ of G, unique up to unitary equivalence; indeed, the theory describes a way of realizing π_ℓ on $\mathcal{L}^2(\mathbf{R}^k)$ for some k. (The mapping $\ell \mapsto \pi_\ell$ is not 1-1.) Our main result in this section is the following theorem, which is equivalent to Theorem 1.1.

THEOREM 4.1. *Let $L \in \mathfrak{U}(\mathfrak{g})$ be transversally elliptic. Assume that there is a nonempty open set $V \subset \mathfrak{g}^*$ such that* Ker $\pi_\ell(L')$ $\neq 0$ *for all $\ell \in V$. Then L is unsolvable.*

Proof. For $\ell = (\xi, \eta)$, consider the form B_η defined in (1.3). We distinguish two cases:

(a) B_η is singular for all $\eta \in \mathfrak{g}_2^*$ (the degenerate case);
(b) B_η is nonsingular for some $\eta \in \mathfrak{g}_2^*$.

In the latter case, G has square integrable representations (see [19]), and B_η is nonsingular for generic η; that is, the generic rank k of B_η is n_1. We shall say in this case that G is *generically* an (H)-group. (See [8].)

We consider case (a) first (where $n_1 - k > 0$). We shall prove the theorem in this case by showing that the hypotheses cannot be satisfied.

We begin by constructing the generic representations in the degenerate case; cf. [23] or [12]. For $\ell \in \mathfrak{g}^*$, $\ell = (\xi, \eta)$, let $B_\ell = B_\eta$; define

$$\mathfrak{g}^*_{\text{reg}} = \{\ell \in \mathfrak{g}^*\colon \text{Rank } B_\ell = k\}.$$

Then $\ell_0 = (\xi_0, \eta_0) \in \mathfrak{g}^*_{\text{reg}}$ iff $(\xi, \eta_0) \in \mathfrak{g}^*_{\text{reg}}$ for every $\xi \in \mathfrak{g}_1^*$. Given $\eta = (0, \eta) \in \mathfrak{g}^*_{\text{reg}}$, let $R_1, \ldots, R_p$ be a basis for the radical of β_η in $\mathfrak{g}_1$. Then, as in [23] or [12], one may choose U_i, $V_i \in \mathfrak{g}_1$, $1 \le i \le k$, such that the R_j, U_i, and V_i $(1 \le j \le p,\ 1 \le i \le k)$ form a basis of $\mathfrak{g}_1$. For $\xi \in \mathfrak{g}_1^*$, one may now model $\pi_{(\xi,\eta)}$ on $\mathcal{L}^2(\mathbf{R}^k)$ so that

$$(4.2)\quad \begin{cases} \text{(i)}\ \pi_{(\xi,\eta)}(R_j) = \sqrt{-1}\,\xi(R_j), & \text{all } j; \\ \text{(ii)}\ \pi_{(\xi,\eta)}(U_i) = \partial/\partial s_i, & \pi_{(\xi,\eta)}(V_i) = \sqrt{-1}\,s_i I,\ \forall i; \\ \text{(iii)}\ \pi_{(\xi,\eta)}|\mathfrak{g}_2 = \sqrt{-1}\,\eta I. \end{cases}$$

We can also arrange matters so that the R_j, U_i, and V_i vary analytically with η on any component of $\mathfrak{g}^*_{\text{reg}}$.

By shrinking V if necessary, we may assume that V is contained in one component $\mathcal{C}$ of $\mathfrak{g}^*_{\text{reg}}$. Let ℓ vary in $\mathcal{C}$, and apply the representations π_ℓ described above.

LEMMA 4.3. *There exist a function μ: $\mathcal{C} \to \mathbf{R}^k$, analytic and homogeneous of degree* 1 *in ξ, and realizations of the π_ℓ, $\ell \in \mathcal{C}$, on $\mathcal{L}^2(\mathbf{R}^k_s)$, such that*

$$(4.4)\qquad \pi_\ell(L) = \sum_{|\alpha|+|\beta|+2|\gamma|\le m} a_{\alpha\beta\gamma}(\eta) s^\alpha (D_s)^\beta \mu^\gamma,$$

with the $a_{\alpha\beta\gamma}$ analytic functions and

$$(4.5)\qquad \sum_{|\alpha|+|\beta|+2|\gamma|=m} a_{\alpha\beta\gamma}(\gamma) s^\alpha t^\beta u^\gamma \ne 0 \quad \text{for} \quad |s|+|t|+|u| \ne 0.$$

Proof. This is essentially contained in Proposition 1.3 of [12], except for the analyticity and homogeneity of μ. The analyticity is clear because the bases can be chosen to be analytic, and the homogeneity in ξ follows from (4.2)(i).

PROPOSITION 4.6. *If G is not generically a type (H)-group, then we cannot have* ker $\pi_\ell(L^\tau) \ne 0$ *for all ℓ in a nonempty open set $V \subset \mathfrak{g}$. (That is, case (a) cannot occur in Theorem* 4.1.)

Proof. Apply Lemma 4.3 to L^τ; then $\pi_\ell(L^\tau)$ has the form of the right-hand side of (4.4). In the proof of Théorème 3.1 of [2], it is shown that for any fixed η_0, there exists K such that if $|\mu| > K$, then the right-hand side of (4.4) is injective. However, the operators $\ell \mapsto \pi_\ell(L^\tau)$ form an analytic family of unbounded operators (as in Section 3); thus one can show as before that if Ker $\pi_\ell(L^\tau) \ne (0)$ in $V \subset \mathcal{C}$, then Ker $\pi_\ell(L^\tau) \ne (0)$ throughout $\mathcal{C}$. But $\mathcal{C}$ is a union of $\mathfrak{g}_1^*$-cosets; thus if $(\xi_0, \eta_0) \in \mathcal{C}$, then one can find $\xi \in \mathfrak{g}^*$ such that $(\xi, \eta_0) \in \mathcal{C}$ and $|\mu(\xi, \eta_0)| > K$. (Here we use the fact that μ is homogeneous of degree 1 in ξ.) This contradicts the assumption that Ker $\pi_{(\xi,\eta_0)}(L^T) \neq (0)$.

We have thus reduced the proof of Theorem (4.1) to case (b), where G is generically an (H)-group. Here, we use a Fourier integral operator to reduce L microlocally to an operator of the form (3.2).

LEMMA 4.7. *Suppose that G is generically an (H)-group. Let $\eta_0 \in \mathfrak{g}_2^*$ be fixed with rank $B_\eta = n_1$. (Here, n_1 must be even.) There exist a Fourier integral operator F associated to a homogeneous canonical transformation χ and a choice of realizations π_η such that*

$$\chi: T^*(G) \to T^*(\mathbf{R}_t^{n_1} \times \mathbf{R}_y^{n_2});$$

$$\chi: (0, 0; 0, \eta_0) \mapsto (0, 0; 0, \eta_0);$$

$$L \sim F^{-1}PF \text{ in a conic neighborhood of } (0, 0; 0, \eta_0),$$

with $P = P(t, D_t, D_y)$, where

$$\pi_\eta(L) = P(t, D_t, \eta).$$

Proof. This is proved in the course of proving Propositions 4.7 and 4.8 of [8].

Since microlocal unsolvability is unchanged under a canonical transformation, Lemma 4.7 and Theorem 3.4 show that L is microlocally unsolvable. Theorem 4.1 now follows from Proposition 2.1.

Part II. A Sufficient Condition for Local Solvability on H-Groups

5. Uniform Semiglobal Solvability of Some Operators. This section and the next are devoted to the proof of Theorem 1.3. As noted in the Introduction, only the implication (i) ⇒ (ii) remains to be proved. We shall actually prove slightly more.

THEOREM 5.1. *Let L be a left invariant operator on a group of type (H) which is elliptic in the generating directions. Suppose that L^T is injective on $\mathcal{L}^2(G)$. Then L is uniformly semiglobally solvable. (See Section 1 for the definition.)*

Proof. By replacing L with LL^*, we may assume that L is self-adjoint and positive. Since $\mathfrak{g}$ is of type (H), $n_1 = \dim \mathfrak{g}_1$ is even; say $n_1 = 2n_0$. We use one other piece of notation: for $f \in \mathcal{L}^2(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$, $f^\wedge$ will be the partial Fourier transform in the second variable.

Let K be a positive number and let $F \in \mathcal{C}_c^\infty(\mathfrak{g}_2^*)$ satisfy

$$F(\eta) = 1 \quad \text{if} \quad |\eta| \leq K;$$

$$F(\eta) = 0 \quad \text{if} \quad |\eta| \geq 2K.$$

Recall that for $\ell = (\xi, \eta) \in \mathfrak{g}^*$ with $\eta \neq 0$, the radical R_ℓ is just $\mathfrak{g}_2$; thus the irreducible unitary representations of G which are nontrivial on $\mathfrak{g}_2$ are square integrable and are parametrized by $\mathfrak{g}_2^*\backslash\{0\}$. Define maps U_1, U_2: $\mathcal{L}^2(G) \to \mathcal{L}^2(G)$ by

$$(5.2) \qquad \pi_\eta(U_1\phi) = F(\eta)\pi_\eta(\phi) \quad \text{for} \quad \eta \neq 0; \qquad U_2 = I - U_1.$$

(More precisely, we define U_1 by (5.2) and the Plancherel theorem for $\phi \in \mathcal{L}^1(G) \cap \mathcal{L}^2(G)$; since $\|U_1\| \leq 1$, U_1 extends to $\mathcal{L}^2(G)$.)

LEMMA 5.3.

(a) *For $\phi \in \mathcal{S}(G)$, $U_1\phi$ and $U_2\phi$ are $\mathcal{C}^\infty$ functions.*
(b) *There are tempered central distributions χ_1, χ_2 such that $U_j\phi = \phi * \chi_j$, $j = 1, 2$.*

Proof. If suffices to prove the lemma for U_1. From the definition, U_1 commutes with left and right translations. Hence if $D \in \mathfrak{U}(\mathfrak{g})$, we have

$$D(U_1\phi) = U_1(D\phi).$$

Therefore $D(U_1\phi) \in \mathcal{L}^2(G)$ for all $\phi \in \mathcal{S}(G)$ and all $D \in \mathfrak{U}(\mathfrak{g})$; (a) now follows via Sobolev theory.

For (b), let $\phi^-(x) = \phi(x^{-1})$, set $\phi_y(x) = \phi(yx)$, and define χ_1 by

$$(\chi_1, \phi) = (U_1\phi^-)(e).$$

This formula makes sense, since $U_1\phi^-$ is smooth. We have

$$\phi * \chi_1(x) = (\chi_1, (\phi_x)^-) = (U_1(\phi_x))(e) = (U_1\phi)_x(e) = (U_1\phi)(x).$$

A similar calculation (using the fact that U_1 commutes with right translation) shows that $\chi_1 * \phi = U_1 \phi$. Hence χ_1 is central.

Now let U be an open set in G with compact closure. We shall establish the following propositions:

PROPOSITION 5.4. *Assume that L satisfies the hypothesis of Theorem 5.1. Then there is a tempered distribution σ such that*

$$L(f*\sigma) = U_2 f \quad \textit{for all} \quad f \in \mathcal{S}(G).$$

PROPOSITION 5.5. *Let L be elliptic in the generating directions. Given the open set U as above, there exists $K > 0$ such that if $g \in \mathcal{L}^2(G)$ satisfies*

$$\hat{g}(x, \eta) = 0 \quad \text{for} \quad |\eta| \geq K,$$

then there is a function $u \in \mathcal{L}^2(G)$ with $Lu = g$ on U.

These propositions together imply Theorem 5.1. To see this, let Δ be a right invariant Laplacian. For any integer $q \geq 0$, there is an integer $p > 0$ such that for any given bounded open set U, there is a function $g \in \mathcal{C}_c^q(G)$ with $\Delta^p g = \delta$ on U. Let σ be as in Proposition 5.4, and use Proposition 5.5 to find $v_2 \in \mathcal{L}^2(G)$ with $Lv_2 = U_2 g$ on U. Then

$$L(\Delta^p(v_2 + g*\sigma)) = \Delta^p(Lv_2 + L(g*\sigma)) = \Delta^p(U_2 g + U_1 g)$$

$$= \Delta^p g = \delta \quad \text{on} \quad U,$$

and we need only choose p so that q is greater than the order of σ as a distribution.

We conclude this section with the proof of Proposition 5.5; Proposition 5.4 will be proved in the next section.

Proof of Proposition 5.5. We follow the method of [16]. Write

$$L_m = P_m = Q_m,$$

where $P_m = P_m(D_x)$ is elliptic on $\mathfrak{g}_1$, and $Q_m = Q_m(D_x, x, D_y)$ is such that $Q_m(D_x, x, 0)$ is constant. (This is possible because L_m is elliptic in the generating directions.) By perhaps adding a constant to P_m, we may assume that $P_m(\xi)$ is always positive. Note also that Q_m and the L_j with $j < m$ are of lower order in D_x than P_m is. Now write

$$L = P_m(D_x) + Q(D_x, x, D_y).$$

Given U, choose a function $\psi_1 \in \mathcal{C}_c^\infty(\mathfrak{g}_1)$ such that $\psi_1(x) = 1$ for all x such that $(x, y) \in U$ for some y; let K' be large enough that $\psi_1(x) = 0$ whenever $|x| \geq K'$. Choose K in a manner to be described below, and let $\psi_2 \in \mathcal{C}^\infty(\mathfrak{g}_1^*)$ be such that $0 \leq \psi_2 \leq 1$, $\psi_2(\xi) = 1$ for $|\xi| \geq 2K$, and $\psi_2(\xi) = 0$ for $|\xi| \leq K$. We know that P_m is invertible in L^2; furthermore, $\psi_2(D_x)$ is a bounded operator on $\mathcal{L}^2(\mathfrak{g})$. Let

$$\mathcal{H}_0 = \{ f \in \mathcal{L}^2(\mathfrak{g}) : \hat{f_2}(x, \eta) = 0 \quad \text{for} \quad |\eta| \geq K \},$$

and define $A = a(D_x, x, \eta)$ by

$$a(\xi, x, \eta) = P_m(\xi)^{-1} \psi_2(\xi) Q(\xi, x, \eta) \psi_1(x).$$

Just as in the proof of Lemma 4.6 of [16], one sees that $A\mathcal{H}_0 \subseteq \mathcal{H}_0$ and that $I + A$ is invertible for suitably large K. Choose K accordingly.

Now let $u_0 = (I + A)^{-1} P_m^{-1} g$, g as in the hypothesis. Then

$$\begin{aligned} Lu_0 &= P_m(I + A)^{-1} P_m^{-1} g + Qu_0 \\ &= g - P_m A (I + A)^{-1} P_m^{-1} g + Qu_0 \\ &= g - \psi_2(D_x) Q \psi_1(x) u_0 + Qu_0 \\ &= g + (I - \psi_2(D_x)) Q \psi_1(x) u_0 + Q(1 - \psi_1(x)) u_0. \end{aligned}$$

The last term is 0 on U, and the middle term is analytic (because its Fourier transform has compact support). Now Proposition 5.5 follows from the Cauchy-Kovalevski Theorem.

6. The Proof of Proposition 5.4. We follow the general outline of the proof of Theorem I in [24]; we shall often refer there for details.

We use spherical coordinates on $\mathfrak{g}_2^*$; a typical element of $\mathfrak{g}_2^* \setminus \{0\}$ is $\eta = (\rho, \omega)$, where $\rho > 0$ and $|\omega| = 1$. Recall that $\dim \mathfrak{g}_2 = n_2$ and $\dim \mathfrak{g}_1 =$

$2n_0$; recall also that there is a 1-1 map $\eta \mapsto \pi_\eta$ of $\mathfrak{g}_2^*\backslash\{0\}$ into $G^\wedge$ given by the Kirillov correspondence. (Thus $\pi_\eta(Y) = i\eta(Y)I$ for all $Y \in \mathfrak{g}_2$.) If we realize $\pi_{(1,\omega)}$ on a Hilbert space $\mathcal{H}$, then we can realize $\pi_{(\rho,\omega)}$ on $\mathcal{H}$ by the equations

$$\pi_{(\rho,\omega)}(X) = \rho^{1/2}\pi_{(1,\omega)}(X), \qquad X \in \mathfrak{g}_1;$$

$$\pi_{(\rho,\omega)}(Y) = \rho\pi_{(1,\omega)}(Y), \qquad Y \in \mathfrak{g}_2.$$

Henceforth we shall always assume that the $\pi_{(\rho,\omega)}$ are realized in this way.

Let $\sigma^{(\lambda,\omega)}(L) = \lambda^m \pi_{(\lambda^{-2},\omega)}(L)$; recall that L has order m. Then

$$\sigma^{(\lambda,\omega)}(L) = \sum_{j=0}^{m} \lambda^{m-j}\pi_{(1,\omega)}(L_j). \tag{6.1}$$

This formula makes formal sense for all $\lambda \in \mathbf{R}$ (positive or not).

Define Sobolev spaces $\mathcal{H}^s(\eta)$, $s = 0, 1, \ldots$ on the representation space $\mathcal{H}(\eta)$ of π_η as follows: $f \in \mathcal{H}^s(\eta)$ if $f \in \operatorname{Dom} \pi_\eta(D)$ for all $D \in \mathfrak{U}_s(\mathfrak{g})$. A norm for $\mathcal{H}^s(\eta)$ is given by

$$\|f\|_s^2 = \sum_{D \in \mathfrak{D}_s} \|\pi_\eta(D)f\|_0^2,$$

where $\| \ \|_0$ is the usual Hilbert space norm on $\mathcal{H}(\eta)$ and $\mathfrak{D}_s$ is some (fixed) basis for $\mathfrak{U}_s(\mathfrak{g})$.

LEMMA 6.2. *Regard the unit sphere in $\mathfrak{g}_2^*$ as S^{n_2-1}, imbedded in $\mathbf{R}^{n_2} \subset \mathbf{C}^{n_2}$. For each $\omega_0 \in S^{n_2-1}$ and each $\epsilon > 0$, there is an open neighborhood V_0 (in $\mathbf{R}^{n_2}$) with the following property: for all $\omega \in V_0$ and all $\rho \geq \epsilon$, $\pi_{(\rho,\omega)}$ can be realized on $\mathcal{L}^2(\mathbf{R}^{n_0})$ so that:*

(a) *$\mathcal{H}^s(\rho, \omega)$ is independent of ρ and ω (so that we may write $\mathcal{H}^s(\rho, \omega) = \mathcal{H}^s$).*

(b) Dom $\sigma^{(\lambda,\omega)}(L) = \mathcal{H}^m$ *for all $\lambda \leq \epsilon^{-2}$ and all $\omega \in V_0$ (recall that $\lambda = K^{-2}$ when $\lambda \neq 0$).*

(c) *The operator $\sigma^{(\lambda,\omega)}$ is a differential operator with polynomial coefficients depending analytically on λ and ω. In particular, there is a neighborhood V of V_0 in $\mathbf{C}^{n_2}$ such that the definition of $\sigma^{(\lambda,\omega)}(L)$ extends formally to $\{\lambda \in \mathbf{C}: |\lambda| < \epsilon^{-2}\} \times V$ and* $\operatorname{Dom}(\sigma^{(\lambda,\omega)}(L)) = \mathcal{H}^s$ *for all such (λ, ω).*

(d) *The operators $\sigma^{(\lambda,\omega)}(L)$ have discrete spectrum, and all eigenvalues have finite multiplicity.*

Note. This lemma says that the $\sigma^{(\lambda,\omega)}(L)$ form a "holomorphic family of type A"; see [10].

Proof. This lemma is proved in Section 4 of [24] in the case where $L = L_m$. Since L_m dominates the lower order terms, the same proof applies here.

We now define the distribution σ. It is shown in [18] that Plancherel measure on $G^\wedge$ is

$$c_0|Pf(\rho,\omega)|\rho^{n_2-1}d\rho d\omega,$$

where $d\omega$ is an invariant measure on S^{n_2-1}, $d\rho$ is Lebesgue measure on $\mathbf{R}$, c_0 is a normalizing constant, and Pf is the Pfaffian of the bilinear form $B_{(\rho,\omega)}$ attached to $(\rho,\omega) = \eta$. If $\pi_\eta(L)$ were invertible for all η with $|\eta| \geq K$, then we could solve $Lu = U_2 f$ as in [21] or [3]: define u by

$$\langle\psi, u\rangle =$$

$$c_0 \int_0^\infty \int_{S^{n_2-1}} \rho^{n_2-1}|Pf(\rho,\omega)|Tr(\pi_{(\rho,\omega)}(L)^{-1}\ \pi_{(\rho,\omega)}(U_2 f)^* \pi_{(\rho,\omega)}(\psi))d\omega d\rho.$$

(Note that $\pi_{(\rho,\omega)}(U_2 f) = 0$ for $|\rho| < K$.) Unfortunately, $\pi_{(\rho,\omega)}$ need not be invertible for all large ρ. However, the set of $\eta = (\rho,\omega)$ for which $\pi_\eta(L)$ is not invertible is the set of zeroes of an analytic function (Theorem VII-1.7 of [10]); if this set contained all η with $|\eta| \geq K$, then we could easily use the Plancherel theorem to construct an element of Ker $L \cap \mathcal{L}^2(G)$. Thus this set is fairly small. To deal with it, we must do some work.

Note that $\pi_{(\rho,\omega)}(L)$ and $\sigma^{(\lambda,\omega)}(L)$, $\lambda = \rho^{-1/2}$, have the same kernel.

We now prove a lemma which enables us to reduce the problem to one involving operators on finite-dimensional spaces.

LEMMA 6.3. *There are orthogonal projections $P_{\rho,\omega}$ on $\mathcal{L}^2(\rho,\omega)$, a $\mathcal{C}^\infty$ function χ, and a constant K_0 such that:*

(a) *$P_{\rho,\omega}$ commutes with $\pi_{\rho,\omega}$;*

(b) *the $P_{\rho,\omega}$ vary analytically on a neighborhood of* supp χ;

(c) *Range $P_{\rho,\omega}$ is finite-dimensional and contains* Ker $\pi_{\rho,\omega}(L)$;

(d) supp χ *is a compact subset of* $[K,\infty] \times S^{n_2-1}$, *and* $\chi(\rho,\omega) = 1$ *if* ker $\pi_{\rho,\omega}(L) \neq (0)$;

(e) *for each $\rho > K$ and ω, $\exists$ an operator $S(\rho,\omega)$ with $\|S(\rho,\omega)\| \leq K_0$ and $S(\rho,\omega)\sigma^{(\rho^{-1/2},\omega)}(L) = I - \chi(\rho,\omega)P_{\rho,\omega}$.*

Proof. This is proved in much the same way as Proposition 3.1 of [24]. Locally, $P_{\rho,\omega}$ is the projection onto the eigenspaces of $\sigma^{(\rho^{-1/2},\omega)}(L)$ with small eigenvalues; one uses a partition of unity to construct χ and to patch the local constructions together. We omit details.

In what follows, we shall assume for notational convenience that the $P_{\rho,\omega}$ all have the same rank. (The rank is constant on connected components of supp ω, and the proof that follows is in essence the same as that for the general case.)

Define σ_2 by

(6.4)

$$(\sigma_2, \phi) = c_0 \int_0^\infty \int_{S^{n_2-1}} \rho^{m+n_2-1} |Pf(\rho, \omega)| Tr(S^\tau(\rho, \omega)\pi_{(\rho,\omega)}(U_2\phi)) d\omega d\rho$$

$$= c_0 \int_K^\infty \int_{S^{n_2-1}} \rho^{m+n_2-1} |Pf(\rho, \omega)| F(\gamma) Tr(S^\tau(\rho, \omega)\pi_{(\rho,\omega)}(\phi)) d\omega d\rho,$$

$\eta = (\rho, \omega)$.

LEMMA 6.5.

(a) *The functional σ_2 is a tempered distribution.*

(b) $$(\sigma_2, L^\tau\phi) = c_0 \int_K^\infty \int_{S^{n_2-1}} \rho^{n_2-1} |Pf(\rho, \omega)| \times Tr(I - \chi(\rho, \omega)P_{\rho,\omega})\pi_{\rho,\omega}(U\phi) d\omega d\rho.$$

Proof. For (a), the analysis in Section 6 of [24] applies, in a simplified form (we need only bound the second integral on the right-hand side in formula (6.1) of [24]). Part (b) is a straightforward computation, using (e) of Lemma 6.3.

Now let $M(\rho, \omega) = \pi_{(\rho,\omega)}(L)|_{Im\, P_{(\rho,\omega)}}$. We need only worry about $M(\rho, \omega)$ on a neighborhood of supp χ. As described in [24], there is a basis $\{e_i^\eta\}$ of $V_\eta = \text{Range } P_\eta$, $\eta = (\rho, \omega)$, such that:

(a) each e_i^η is a $\mathcal{C}^\infty$ vector;

(b) $\eta \mapsto \langle e_i^\eta, w\rangle$ is locally analytic for each $w \in \mathcal{L}^2(\mathbf{R}^{n_2})$ when the π_η are realized as in Lemma 6.2;

(c) $M(\rho, \omega)$ is given locally by an analytic matrix $(A_{ij}(\eta)) = A(\eta)$, where $\eta = (\rho, \omega)$.

Let $B(\eta)$ be the cofactor matrix of $A(\eta)$, so that when $\operatorname{Det} A(\eta) \neq 0$,

$$A(\eta)^{-1} = (\operatorname{Det} A(\eta))^{-1} B(\eta).$$

Set $\Omega = \operatorname{supp} \chi \cap ([K, \infty] \times S^{n_2-1})$, and say that $\psi \in \mathcal{C}_0^\infty(\Omega)$ if ψ is defined on Ω, ψ vanishes at ∞, and ψ extends to a $\mathcal{C}^\infty$ function that is 0 off Ω.

LEMMA 6.6. *There is a distribution $\tilde{A}$ of finite order such that if $\psi \in \mathcal{C}_0^\infty(\Omega)$, then*

$$\tilde{A}(A(\eta)\psi(\eta)) = c_0 \int_K^\infty \int_{S^{n_2-1}} \rho^{n_2-1}\, \psi(\rho, \omega)\, d\omega\, d\lambda.$$

Proof. Let $(\rho, \omega) = \Phi(\kappa, \omega) = (\kappa^{-1/2}, \omega)$; Φ maps $[K, \infty] \times S^{p-1}$ to $[0, K^{-1/2}] \times S^{p-1}$, and $(\operatorname{Det} A) \circ \Phi^{-1}$ extends to an analytic function on a neighborhood of $\Phi(\Omega)$ in $\mathbf{R} \times S^{n_2-1}$. Now the same analysis as in Section 4 of [24] (using [1] or [15], plus a partition of unity) shows that $\rho^{2n_2-1} A \circ \Phi^{-1}$ has a distribution inverse A' of finite order. Define

$$\tilde{A} = \frac{n_2}{2n_2 - 1} (A' \circ \Phi) \quad \text{on} \quad \mathcal{C}_0^\infty(\Omega);$$

$\tilde{A}$ has the desired property, as an easy calculation (using the change of variables formula) shows.

Now define σ_1 by

$$(\sigma_1, \phi) = c_0 \int_0^\infty \int_{S^{n_2-1}} \rho^{n_2-1}\, \chi(\rho, \omega)$$
$$\times \sum_j \tilde{A}(\langle \pi_{\rho,\omega}(U_2\Phi) e_j^{(\rho,\omega)}, B^\tau(\rho, \omega) e_j^{(\rho,\omega)} \rangle)(Pf(\rho,\omega))\, d\omega\, d\rho.$$

LEMMA 6.7.

(a) *The functional σ_1 is a tempered distribution.*

(b) $(\sigma_1, L^\tau\phi) = c_0 \displaystyle\int_0^\infty \int_{S^{n_2-1}} \rho^{n_2-1}\, Tr(\chi(\rho, \omega) P(\rho, \omega) \pi_{\rho,\omega}(U\phi))$

$$\times\ |Pf(\rho, \omega)|\, d\omega\, d\rho.$$

Proof. For each j, the map

$$\phi \mapsto \langle \pi_{\rho,\omega}(U_2\phi)e_j^{(\rho,\omega)}, B^\tau(\rho, \omega)e_j^{(\rho,\omega)}\rangle \chi(\rho, \omega)$$

is a continuous map from $\mathcal{S}(G)$ to $\mathcal{C}_c^\infty(\Omega)$. (The relevant estimates are like those in Section 6 of [24].) Now Lemma 6.6 applies.

(b) The calculations are like those for equations (3.9) and (3.10) in [24], but somewhat simpler, we omit details.

Now let $\sigma = \sigma_1 + \sigma_2$. From (a) of Lemmas 6.5 and 6.7, σ is a tempered distribution; from (b) of those lemmas,

$$\begin{aligned}(L\sigma, \phi) = (\sigma, L^\tau\phi) &= c_0 \int_0^\infty \int_{S^{n_2-1}} \lambda^{n_2-1}|Pf(\lambda, \omega)|\, Tr(\pi_{\lambda,\omega}(U\phi))\,d\omega\, d\lambda \\ &= (U\phi)(e).\end{aligned}$$

This completes the proof of Proposition 5.4.

Part III. The Existence of Global Fundamental Solutions

Let G be any Lie group, with Lie algebra $\mathfrak{g}$. Recall that in Section 1 we defined the left invariant differential operator L to be uniformly semiglobally solvable of order $\leq r$ (where $r \geq 0$) if there exist an increasing family of open sets Ω_m in G with compact closure and a sequence of distributions ξ_m such that

(a) $\bigcup_{m=1}^{\infty} \Omega_m = G$;

(b) ξ_m is of order $\leq r$;

(c) $L\xi_m|_{\Omega_m^-} = \delta =$ unit mass at the identity of G.

Choose an ordered basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{g}$; let $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index. The Poincaré-Birkhoff-Witt Theorem says that the elements X^α form a basis for $\mathfrak{U}(\mathfrak{g})$. Set $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If k is a nonnegative integer, let $|\phi|_{k,m} = \sup_{|\alpha|\leq k, x\in\Omega_m} |X^\alpha\phi(x)|$ for $\phi \in \mathcal{C}_c^\infty(\Omega_m^-)$; let $\|\ \|_{k,m}$ be the dual norm for the distributions of order $\leq k$ on Ω_m^-.

The main result of this section is the following:

THEOREM 7.1. *Let L be a left invariant differential operator on the Lie group G which is uniformly semiglobally solvable of order $\leq r$. Sup-*

pose that G is L-convex (see [6]). Then L has a fundamental solution of bounded order.

Before giving the proof, we note two corollaries. Recall ([6], Lemmas 2, 3) that if G is nilpotent, it is L-convex.

COROLLARY 7.2. *If L is a locally solvable homogeneous left invariant operator on the stratified nilpotent Lie group G (where G is simply connected) then L has a fundamental solution of order $\leq r$.*

Proof. Let $|\ |$ be a homogeneous norm on G, and let $\{\alpha_t : t > 0\}$ be the 1-parameter family of dilations on G. If L is locally solvable, then L has a local fundamental solution ξ on some ball about the identity (see [22]). By applying the homogeneous dilations of G to ξ, it is easy to show that L has local fundamental solutions of the same order on every ball. (See [5] for details.) Now Theorem 7.1 applies.

COROLLARY 7.3. *Let L be a transversally elliptic left invariant operator on the connected, simple connected H-group G. If L is locally solvable, then L has a fundamental solution.*

Proof. This is immediate from Theorems 5.1 and 7.1.

The proof of Theorem 7.1 is along the same lines as the proof of Theorem 3.5.5 in [9], but there are some differences. In what follows, we assume that $\Omega_m = \Omega_m^{-1}$, all m. We begin with a simple lemma.

LEMMA 7.4. *Suppose that L is uniformly globally solvable of order $\leq r$. Then there is a sequence of numbers $\{K_m\}$ such that for every $f \in \mathcal{C}_c^\infty(\Omega_m^-)$, there is a function $u_m \in \mathcal{C}_c^\infty(\Omega_{m+1})$ such that*

(a) $Lu_m = f$ *on* Ω_m;
(b) $\|u_m\|_{r,m} \leq K_m \|f\|_{0,m}$.

Proof. Let $\psi_\mu \in \mathcal{C}_c^\infty(\Omega_{\mu+1})$ be identically 1 on Ω_m; let $u_m(x) = \psi_m(x)(f * \xi_{m'})(x)$, where m' is such that $\Omega_m^3 \subset \Omega_{m'}$. The map $f \mapsto u_m$ is continuous from the $\|\ \|_{0,m}$ norm to the $\|\ \|_{r,m}$ norm; let K_m be a bound for the norm of this map. Now the conditions of the lemma are easily verified.

Proof of Theorem 7.1. Let the K_m be as in Lemma 7.4, and let $\{\epsilon_m\}$ be a decreasing sequence of positive numbers such that $\Sigma_{m=1}^\infty \epsilon_m(1 + K_m)$ converges. We shall find functions $u_m, f_m \in \mathcal{C}_c^\infty(\Omega_{m+1})$, $m = 1, 2, \ldots,$ such that:

$$Lu_m = f_m; \tag{7.5}$$

(7.6) $$\|f_m - \delta\|_{s,m} < \epsilon_\mu;$$

(7.7) $$\|u_{m+1} - u_m\|_{r_0,m} < 2^{-m} + 2K_m\epsilon_m.$$

Then (7.7) implies that the u_m are converging on each Ω_k to a distribution of order $\leq r_0$; hence the u_m are converging in $\mathfrak{D}'(G)$ to a distribution ξ of order $\leq r_0$. Since $Lu_m = f_m \to \delta$ (from (7.5) and (7.6)), we see that $L\xi = \delta$.

We define the u_m, f_m inductively. For $m = 1$, we let $\tilde{f}_1$ approximate δ closely enough to satisfy (7.6), find u_1 by using Lemma 7.4, and set $f_1 = Lu_1$; since $f_1 = \tilde{f}_1$ on Ω_1, (7.6) still holds. Now assume that we have u_m, f_m. Choose $g_m \in \mathcal{C}_c^\infty(\Omega_m)$ such that

$$\|\delta - f_m - g_m\|_{s,m} < \epsilon_{m+1}/2.$$

Then, of course,

$$\|g_m\|_{0,m} < \epsilon_{m+1}/2 + \|\delta - f_m\|_{s,m} < \epsilon_{m+1} + \epsilon_m < 2\epsilon_m.$$

Let $v_m \in \mathcal{C}_c^\infty(\Omega_{m+1})$ satisfy $Lv_m = g_m$ on Ω_m, $\|v_m\|_{r_0,m} \leq K_m\|g_m\|_{s,m}$. (This is possible by Lemma 7.4.) Then

(7.8) $$\|v_m\|_{r_0,m} \leq 2K_m\epsilon_m$$

and

$$L(u_m + v_m) = f_m + L(v_m) = f_m + G_m,$$

where G_m $(= L(v_m)) = g_m$ on Ω_m. Therefore

(7.9) $$\|\delta - f_m - G_m\|_{s,m} < \frac{1}{2}\epsilon_{\mu+1}.$$

Now choose $F_{m+1} \in \mathcal{C}_c^\infty(\Omega_{m+1})$ such that

$$F_{m+1}|_{\Omega_m} = (f_m + G_m)|_{\Omega_m};$$

(7.10) $$\|F_{m+1} - \delta\|_{s,m+1} < \epsilon_{m+1}.$$

(Since $\| \|_{s,m+1}$ is the total variation norm on measures, (7.9) implies that the inequality (7.10) can be satisfied.) Now set $h_{m+1} = F_{m+1} - f_m - G_m$. The semiglobal solvability of L (or Lemma 7.4) implies that we can find $w_{m+1} \in \mathcal{C}_c^\infty(\Omega_{m+2})$ such that $Lw_{m+1} = h_{m+1}$ on Ω_{m+1}. Since h_{m+1} is 0 on Ω_m and G is L-convex, we can find a function $W_{m+1} \in \mathcal{C}_c^\infty(\Omega_{m+2})$ such that $LW_{m+1} = 0$ on Ω_{m+1} and

$$\| W_{m+1} - w_{m+1} \|_{r_0,m} < 2^{-m}. \tag{7.11}$$

(See pp. 393–394 of [25] for a proof.)

Now set $u_{m+1} = u_m + v_m + w_{m+1} - W_{m+1}$, $f_{m+1} = Lu_{m+1}$. Then $\operatorname{supp} u_{m+1} \subset \Omega_{m+2}$, and u_{m+1}, f_{m+1} satisfy (7.5). On Ω_{m+1},

$$\begin{aligned} f_{m+1} &= L(u_m + v_m) + L(w_{m+1}) - L(W_{m+1}) \\ &= f_m + G_m + (F_{m+1} - f_m - G_m) = F_{m+1}; \end{aligned}$$

in view of (7.10), f_{m+1} satisfies (7.6). Finally,

$$\begin{aligned} \| u_{m+1} - u_m \|_{r_0,m} &\le \| v_m \|_{r_0,m} + \| W_{m+1} - w_{m+1} \|_{r_0 m} \\ &< 2^{-m} + 2\epsilon_m K_m \quad \text{(from (7.11) and (7.8))}, \end{aligned}$$

so that (1.3) holds. This completes the induction and the proof of Theorem 1.1.

Notes. 1. Essentially the same proof shows that if $f \in \mathcal{D}'(G)$ has order $\le k$, then there is a distribution u with order $\le k + r_0$ such that $Lu = f$. (Of course, L and G satisfy the other hypotheses of Theorem 7.1.)

8. Remarks and Examples. We begin with a simple example illustrating Theorem 1.1.

Example 8.1. Let G be the 5-dimensional Heisenberg group, with

$$[X_1, Y_1] = [X_2, Y_2] = T,$$

and all other brackets 0 (unless given by anticommutativity). For $\eta \ne 0$, one can realize π_η on $L^2(\mathbf{R}^2)$ by

$$\pi_\eta(X_j) = \frac{\partial}{\partial t_j}, \qquad \pi_\eta(Y_j) = i\eta t_j \; (j = 1, 2), \qquad \pi_\eta(T) = i\eta I.$$

Now let $L = \Sigma_{j=1}^{2} (X_j + iY_j + I)(X_j - iY_j)$. Then L is transversally elliptic, and

$$f_\eta(t_1, t_2) = e^{-1/2\eta(t_1^2+t_2^2)} \in \operatorname{Ker} \pi_\eta(L) \quad \text{for} \quad \eta > 0.$$

Hence L is not locally solvable.

It is obviously tempting to try to extend Theorems 1.1 and 1.2 to larger classes of groups. Because sufficient conditions for solvability generally seem to be easier to find than necessary ones, and because our results on sufficiency are weaker than those on necessity, we begin by examining the sufficiency proof.

The proof of Theorem 5.1 naturally divides into the proof of Proposition 5.4 ("solvability near ∞ in $\hat{\mathfrak{g}}_2$") and that of Proposition 5.5 ("solvability near 0"). For Proposition 5.5, the key step is to establish that the $\pi_\eta(L)$ form an analytic family of operators for large η. For this purpose, one needs to know that every representation corresponding to a large value of η is in "general position" (in the sense of Part II, Section 5 of [20]). It was here that we first used the hypothesis that G is of type (H).

One might try to prove a corresponding result for n-step groups. Let G be a stratified Lie group with Lie algebra $\mathfrak{g} = \oplus_{j=1}^{k} \mathfrak{g}_j$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ ($\mathfrak{g}_j = (0)$ if $j > k$). We might say that G is a *generalized (H)-group* if for every nonzero element $\eta \in \mathfrak{g}_k^*$, there is a unique representation π_η of G such that $\pi_\eta(X_k) = i\eta(X_k)$ for all $X_k \in \mathfrak{g}_k$. It is easy to check that the π_η are all square integrable and constitute the elements of $G^\wedge$ in general position (if $k > 1$). When $k = 2$, generalized (H)-groups are just the (H)-groups.

For some of these groups, unfortunately, the Sobolev spaces $\mathcal{H}^s(\pi_\eta)$ do not seem to be independent of η. (An example is given below.) Accordingly, we may need to restrict this class of groups further. There are at least two subclasses for which the Sobolev spaces are independent of η: those with 1-dimensional center (e.g., those constructed in [4]), and those of step 3. For these groups, it should not be hard to prove an appropriate analogue of Proposition 5.4. There remains the problem of "solvability at 0"; as of now, we do not have a proof of Proposition 5.5 for these other groups.

We now give the example promised above.

Example 8.2. Let $\mathfrak{g}$ be the 5-step, 10-dimensional Lie algebra spanned by V_j, W_j, X_j, Y_j, and Z_j ($j = 1, 2$), with

$$[V_1, V_2] = W_1, [V_1, W_j] = X_j \ (j = 1, 2), [V_1, X_1] = Y_2, [V_1, X_2] = Y_1,$$

$$[V_1, Y_j] = Z_j \ (j = 1, 2), [V_2, Y_1] = -Z_2, [V_2, Y_2] = Z_1,$$

$$[W_1, X_j] = Z_j \ (j = 1, 2), [W_2, X_1] = -Z_2, [W_2, X_2] = Z_1,$$

and other brackets of basis vectors 0 (unless given by antisymmetry). It is a tedious exercise to verify that $\mathfrak{g}$ is a stratified Lie algebra, with $\mathfrak{g}_1 = \text{span}\{V_1, V_2\}, \ldots, \mathfrak{g}_5 = \text{span}\{Z_1, Z_2\}$. It is then easy to check that $\mathfrak{g}$ is a generalized (H)-group. If ℓ is nontrivial on $\mathfrak{g}_5$, the subalgebra $\mathfrak{h} = \mathfrak{g}_3 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_5$ is polarizing. Let G be the (connected, simply connected) nilpotent Lie group with Lie algebra $\mathfrak{g}$, and set $H = \exp \mathfrak{h}$; then the set

$$S = \{\exp w_1 W_1 \exp W_2 W_2 \exp v_1 V_1 \exp v_2 V_2 \colon w_1, w_2, v_1, v_2 \in \mathbf{R}\}$$

in a cross-section for $H \backslash G$, and we may realize π_ℓ on $\mathcal{L}^2(\mathbf{R}^4)$ by identifying S with $\mathbf{R}^4$ in the obvious way. Then

$$\pi_\ell(V_1) = \frac{\partial}{\partial v_1} - v_2 \frac{\partial}{\partial w_1} + i v_1 v_2 \ell(X_1) + i v_1^2 v_2 \ell(Y_1)/2$$

$$+ i(v_1 v_2 w_1 - v_1^3 v_2/4)\ell(Z_1) + i v_1 w_1 w_2 \ell(Z_2),$$

$$\pi_\ell(V_2) = \frac{\partial}{\partial v_2}.$$

Since π_ℓ depends (up to equivalence) only on $\ell|_{\mathfrak{g}_5}$, it is natural to take $\ell(X_j) = \ell(Y_j) = 0$. In this (rather natural) set of realizations for π_ℓ, the operators $\pi_\ell(V_1)$, $\pi_\ell(V_2)$ span spaces depending on ℓ, and thus $\mathcal{H}^1(\pi_\ell)$ depends on ℓ. It may be that one can find a realization for the π_ℓ such that the Sobolev spaces are independent of ℓ, but this example does show that new techniques will be involved in extending Proposition 5.4.

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