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Transversal Lie group actions on abstract CR manifolds

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0. Introduction and main results

Let M be a manifold of class C^∞ with a CR structure; i.e., there exists a bundle $\mathscr{V}$, called the CR bundle, with $\mathscr{V} \subset \mathbb{C}TM$, the complexified tangent bundle of M, satisfying

$$\mathscr{V} \cap \bar{\mathscr{V}} = (0) \quad \text{and} \quad [\mathscr{V}, \mathscr{V}] \subset \mathscr{V}. \tag{0.1}$$

We shall say that M is of codimension l if $\dim_{\mathbb{C}} \mathscr{V} = n$ and $\dim_{\mathbb{R}} M = 2n + l$.

(0.2) **Definition.** A CR manifold M has a CR *local group action* if there is a Lie group G with an open neighborhood V of the identity e in G and a smooth mapping

$$H: M \times V \to M, \tag{0.3}$$

satisfying the following. First, H is a group action i.e.

$$H(H(m, g_1), g_2) = H(m, g_1 g_2), \quad H(m, e) = m$$

for all $m \in M$ and all g_1, $g_2 \in V$ with $g_1 g_2 \in V$. Second, for each $g \in V$ the mapping $m \mapsto H(m, g)$ is a CR self mapping of M; i.e., its differential maps the CR bundle $\mathscr{V}$ of M into itself.

If M has a group action (Definition (0.2)) and if $X \in \mathfrak{g}$, the Lie algebra of G, we define a vector field X^+ on M by taking X^+_m to be the push forward of X at (m, e) under the mapping H i.e.

$$X^+ f(m) = \frac{d}{dt} f(H(m, \exp tX))|_{t=0} \tag{0.4}$$

for $f \in C^\infty(M)$. We let $\mathscr{E} = \mathscr{E}(G)$ be defined by

$$\mathscr{E} = \{X^+ : X \in \mathfrak{g}\}. \tag{0.5}$$

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A space E of real, smooth vector fields on M is *transversal* to $\mathscr{V}$ if for all $m \in M$

(0.6) $$\mathscr{V}_m \oplus \bar{\mathscr{V}}_m + \mathbb{C}E_m = \mathbb{C}T_m M;$$

it is called *strictly transversal* if $\dim_{\mathbb{R}} E = \dim_{\mathbb{R}} E_m$ for all $m \in M$ and

(0.7) $$\mathscr{V}_m \oplus \bar{\mathscr{V}}_m \oplus \mathbb{C}E_m = \mathbb{C}T_m M$$

(which implies that E is a bundle of dimension l, the codimension of M). A group action is called transversal (resp. strictly transversal) it its $\mathscr{E}(G)$ defined by (0.5) is transversal (resp. strictly transversal) to $\mathscr{V}$.

A real sbmanifold M' of $\mathbb{C}^{n+l}$ of dimension $2n+l$ is *generic* if it is locally defined by $\{Z:\rho_j(Z,\bar{Z})=0, j=1,\ldots,l\}$, ρ_j real, smooth and $\partial\rho_1,\ldots,\partial\rho_l$ linearly independent. It has a natural CR structure with $\mathscr{V}$ being the bundle of antiholomorphic tangent vectors to M'. A CR manifold M is called *embeddable* if there is a CR diffeomorphism between M and a generic submanifold of $\mathbb{C}^{n+l}$.

We have the folowing results.

Theorem 1. *Let M be a* CR *manifold with a local* CR *group action given by a Lie group G. Then $\mathscr{E} = \mathscr{E}(G)$ is a Lie algebra, and*

(0.8) $$[\mathscr{E}, \mathbb{L}] \subset \mathbb{L}$$

where $\mathbb{L} = C^\infty(M, \mathscr{V})$. Conversely, if *$E$ is a finite dimensional Lie algebra of vector fields on M satisfying* (0.8) *then there is a Lie group G and a local* CR *action H of G on any open relatively compact submanifold $\tilde{M} \subset M$* such that *$E = \mathscr{E}(G)$. In addition, G can be chosen so that* $\dim G = \dim E$, *in which case G and H are unique up to group isomorphism.*

Moreover, if M has a transversal local CR *group action, then M is locally embeddable as a generic manifold in $\mathbb{C}^{n+l}$.*

Theorem 2. *Let M be a* CR *manifold with a transversal* CR *local group action H given by a Lie group G, with $\mathscr{E}$ defined by* (0.5). *If* $\dim \mathscr{E}_m$ *is independent of $m \in M$ (i.e. $\mathscr{E}$ defines a subbundle of TM) then there exists a local* CR *embedding π of M into $M' \subset \mathbb{C}^{n+l}$ for which the mapping*

(0.9) $$p \to \pi \circ H(\pi^{-1}(p), g), \quad p \in M'$$

extends holomorphically to an open neighborhood of M' in $\mathbb{C}^{n+l}$ for all $g \in V$. Furthermore, any other embedding with this property is biholomorphically equivalent to π.

(0.10) *Remark.* In Theorem 2, M' need not be a real analytic generic manifold (as the proof of Theorem 2 shows.)

(0.11) *Remarks.* Even for strictly transversal group actions, there are CR embeddings for which the action does not extend holomorphically, as Example (2.22) shows.

To state our final result we recall some standard definitions. If M is a CR manifold and f a function (or distribution) defined on M, we shall say that f is CR if $Lf = 0$ for all $L \in \mathbb{L} = C^\infty(M, \mathscr{V})$, where $\mathscr{V}$ is the CR bundle of M. If M is a generic manifold of $\mathbb{C}^{n+l}$ given by $\rho=0$, with $\rho=(\rho_1,\ldots,\rho_l)$, we define a *wedge of edge* M by

$$\mathscr{W}_\Gamma=\{Z\in\mathcal{O}:\rho(Z,\bar{Z})\in\Gamma\},$$

where $\mathcal{O}$ is an open neighborhood in $\mathbb{C}^{n+l}$ of $p_0\in M$ and Γ is an open cone in $\mathbb{R}^l$. If h is holomorphic in $\mathscr{W}_\Gamma$ with tempered growth near M then its boundary value on M, bh, is a CR distribution (see for example [1] and [3]).

Theorem 3. *Let M be a generic manifold with a strict local group action which extends holomorphically to a neighborhood of M in $\mathbb{C}^{n+l}$. Then locally every* CR *distribution f can be decomposed*

$$f=\sum_{j=1}^{r} bh_j, \tag{0.12}$$

where h_j is holomorphic in a wedge $\mathscr{W}_{\Gamma_j}$ with edge M.

Historically transversal CR group actions were first considered by Tanaka [12], who studied hypersurfaces with one parameter transversal group action. Theorem 1 extends and complements results obtained by the authors jointly with F. Treves [6]. The proof of the embeddability makes use of a theorem proved in [4] and is related to earlier work of Jacobowitz [7], in the abelian case.

Theorem 3 was proved in the case of an abelian group action in [6]. An example of Trépreau [13] shows that decomposition of the form (0.12) does not hold in general (see also [5]). Our proof is based on a mini-FBI transform adapted to the group structure (similar to the one used in [1] in the abelian case) which might be of independent interest.

We would like to thank the referee for pointing out that Lemma (2.1) was incorrectly stated in an earlier version of this paper.

1. CR group actions, proof of Theorem 1

We assume first that we have a local CR group action H on M given by (0.3) and put $\mathscr{E}=\mathscr{E}(G)$ as in (0.5).

Proof of Theorem 1. To show that $\mathscr{E}$ is a Lie algebra and $X\mapsto X^+$ is a homomorphism we refer to Nomizu [10, Chap. 1, Sect. 6]. In order to prove (0.8), let $L\in\mathbb{L}$ and $X^+\in\mathscr{E}$. Then by Nomizu [10, Chap. 1, Sect. 2],

$$[X^+,L]=\lim_{t\to 0}\frac{1}{t}(L-\varphi_t'\cdot L) \tag{1.1}$$

where φ_t is the 1-parameter group of transformations associated to X^+. On the other hand, by a uniqueness argument for ordinary differential equations it is easy to see that

$$\varphi_t(m)=H(m,\exp tX), \tag{1.2}$$

where $\exp:\mathfrak{g}\to G$ is the usual exponential map. Since by definition of a CR map,

$$H'(g,\cdot)(\mathscr{V}_m)\subset\mathscr{V}_{H(m,g)} \tag{1.3}$$

the desired conclusion (0.8) then follows from (1.1), (1.2) and (1.3).

For the second part of Theorem 1, let E be a finite dimensional Lie algebra of vector fields on M and let G be a Lie group with Lie algebra $\mathfrak{g}$ isomorphic to E and

$$\sigma:\mathfrak{g}\to E$$

an isomorphism. We define for $\tilde{M}$ an open relatively compact subset of M and V, a small neighborhood of the identity in G, a mapping $H:\tilde{M}\times V\to\tilde{M}$ as follows. If $X\in\mathfrak{g}$ and t sufficiently small we set

$$H(m,\exp tX)=f(m,t), \tag{1.4}$$

where f is the unique solution of

$$\frac{df}{dt}(m,t)=\sigma(X)(f(m,t)),\quad f(m,0)=m. \tag{1.5}$$

This defines a local group action since the Baker–Campbell–Hausdorff formula (see e.g. [14]) holds (not only in Taylor series) for the exponentials of vector fields which form a finite dimensional Lie algebra; it is clear that $X^+=\sigma(X)$, with X^+ as defined by (0.4). Hence the restriction of E to $\tilde{M}$ is $\mathscr{E}(G)$.

We must show that the action defined by (1.4) is CR, i.e. (1.3) holds. For this, let $L_1,\ldots,L_n$ be a local basis of $\mathbb{L}$ near $m_0\in\tilde{M}$. By assumption (0.8) for every $X^+\in E$ we have

$$[X^+,L_j]=\sum_{k=1}^{n}\alpha_{jk}L_k, \tag{1.6}$$

where α_{jk} are smooth functions defined near m_0. If f is given by (1.4) we write $f_t(m)=f(m,t)$. We must show there exist functions $C_{jk}(m,t)$ so that

$$f_t'(L_{j,f_t^{-1}(m)})=\sum_k C_{jk}(m,t)L_{k,m}, \tag{1.7}$$

i.e. for any smooth function ψ on $\tilde{M}$ near m_0

$$[L_j(\psi\circ f_t)](f_t^{-1}(m))=\sum_k C_{jk}(m,t)L_k\psi(m). \tag{1.8}$$

To prove (1.8) we set

$$u_j(t)=L_j(\psi\circ f_t)(f_t^{-1}(m)),\quad 1\leqslant j\leqslant n. \tag{1.9}$$

By Nomizu [10, Proposition of Sect. 2, Chap. 1] we have

$$\frac{du_j}{dt}(0)=([X^+,L_j]\psi)(m). \tag{1.10}$$

We shall compute $(du_j/dt)(t)$ and show that the u_j satisfy a linear system of ordinary differential equations. Since $f_{t+t'}=f_t\circ f_{t'}$ and hence $f_{t+t'}^{-1}=f_{t'}^{-1}\circ f_t^{-1}$ we have, using the definition (1.9),

$$u_j(t+t')=[L_j(\tilde{\psi}\circ f_{t'})](f_{t'}^{-1}(\tilde{m})) \tag{1.11}$$

where $\tilde{\psi}=\psi\circ f_t$, $\tilde{m}=f_t^{-1}(m)$. Differentiating (1.11) with respect to t', putting $t'=0$

and using (1.10) yields

$$\frac{du_j}{dt}(t) = ([X^+, L_j]\tilde{\psi})(\tilde{m}) \tag{1.12}$$

so that by (1.6)

$$\frac{du_j}{dt}(t) = \sum \alpha_{jk}(f_t^{-1}(m))u_k(t). \tag{1.13}$$

We write (1.13) as

$$\frac{du}{dt}(t) = A(m,t)u(t), \tag{1.14}$$

where $u(t) = (u_1(t), \ldots, u_n(t))$ and $A(m,t)$ is an $n \times n$ matrix with smooth coefficients. For m fixed let $V_1(m,t), \ldots, V_n(m,t)$ be n linearly independent solutions of (1.14). Then we have

$$u(t) = C_1(m)V_1(m,t) + \cdots + C_n(m)V_n(m,t). \tag{1.15}$$

Since $u(0) = (L_1\psi(m), \ldots, L_n\psi(m))$, we conclude that (1.8) follows with C_{jk} smooth and independent of ψ. This completes the proof of (1.3).

Uniqueness of G and the action H, up to group isomorphism, is a consequence of the following lemma.

(1.16) **Lemma.** *Let M be a smooth manifold with two group actions*

$$H^j: V_j \times M \to M, \quad j = 1,2 \tag{1.17}$$

with V_j a neighborhood of the identity in a Lie group G_j. Let $\mathscr{E}_j = \mathscr{E}(G_j)$ be defined by (0.5) *and assume* $\dim G_j = \dim \mathscr{E}_j$, $j = 1,2$. *Then*

$$\mathscr{E}_1 = \mathscr{E}_2 \tag{1.18}$$

if and only if there exists a group isomorphism $\pi: G_1 \to G_2$ such that

$$H^1(m,g) = H^2(m,\pi(g)) \tag{1.19}$$

for $g \in V_1$, $g \in \pi^{-1}(V_2)$, $m \in M$, where V_j is an open neighborhood of the identity in G_j, $j = 1,2$.

Proof. Let $\sigma_j: \mathfrak{g}_j \to \mathscr{E}_j$ be the linear mappings from the Lie algebras $\mathfrak{g}_j$ given by (0.4). By the dimension assumption and the first part of the theorem, the mappings σ_j are Lie algebra isomorphisms. If (1.19) holds we have $\sigma_1 = \sigma_2 \circ \pi'$ so that $\sigma_1(X) = \sigma_2(\pi'(X))$ which proves (1.18).

Conversely, assume (1.18). After a group isomorphism $\pi: G_1 \to G_2$ we may assume $G_1 = G_2 = G$ and $\sigma_1(X) = \sigma_2(X) = X^+$ for all $X \in \mathfrak{g}$. We shall show that $H^1 = H^2$. Let $X \in \mathfrak{g}$, $m \in M$ and put

$$f^j(m,t) = H^j(m, \exp tX), \quad j = 1,2. \tag{1.20}$$

It suffices to show that $f^1(m,t) = f^2(m,t)$ for $|t|$ small. By definition of σ_j we have for φ smooth on M

$$(1.21) \qquad \sigma_j(X)\varphi(m) = \frac{d}{dt}\varphi(f^j(m,t))|_{t=0}, \quad j=1,2.$$

We claim

$$(1.22) \qquad \frac{\partial f^j}{\partial t} = X^+(f^j(m,t)).$$

Indeed,

$$(1.23) \qquad f^j(m,t+t') = H^j(m,\exp(t+t')X) = H^j(H^j(m,\exp tX),\exp t'X).$$

Differentiating (1.23) with respect to t' and setting $t'=0$ then yields (1.22). Since $f^j(m,0)=m$, $j=1,2$, by uniqueness we conclude $f^1=f^2$, which proves Lemma (1.16), and hence the uniqueness in Theorem 1.

For the last statement of Theorem 1, we apply the first part and the main theorem in [4] to conclude the desired local embeddability of M.

(1.24) *Example.* We shall construct a five dimensional CR manifold M, of codimension 3, on which the 3-dimensional Heisenberg group G acts strictly transversally. M is given as embedded in $\mathbb{C}^4$ by the parametrization

$$(1.25) \qquad \Psi:(z,s)\in\mathbb{C}\times\mathbb{R}^3\mapsto(z,s_1+i|z|^2,s_2+i|z|^4,s_3-2is_1|z|^4+2is_2|z|^2).$$

A basis for $\mathbb{L}$ is given by

$$(1.26) \qquad L = \frac{\partial}{\partial\bar z} - izS_1 - 2iz^2\bar zS_2 - 2z^3\bar z^2S_3,$$

where $S_1 = \dfrac{\partial}{\partial s_1} + 2s_2\dfrac{\partial}{\partial s_3}$, $S_2 = \dfrac{\partial}{\partial s_2} - 2s_1\dfrac{\partial}{\partial s_3}$, $S_3 = \dfrac{\partial}{\partial s_3}$. Note that $\{S_1,S_2,S_3\}$ is a basis for the space of right invariant vector fields on the 3-dimensional Heisenberg group with multiplication

$$(1.27) \qquad s\cdot s' = (s_1+s_1', s_2+s_2', s_3+s_3'+2s_1s_2'-2s_1's_2).$$

(In [6] it was proved that if there is a Lie algebra E of vector fields strictly transversal to $\mathscr{V}$, then one can find coordinates (z,s) for M and a basis $\{L_j\}$ for $\mathbb{L}$ of the form $L_j = \dfrac{\partial}{\partial\bar z_j} + \sum a_{jk}(z,\bar z)S_k$, where $\{S_k\}$ is a basis for the right invariant vector fields on the Lie group corresponding to E). If we denote by z, $w=(w_1,w_2,w_3)$ the coordinates in $\mathbb{C}^4$ then the group $G=\mathbb{R}^3$ acts on M through the parametrization (1.25):

$$(1.28) \qquad H((z,w),s') = \psi(z,s\cdot s')$$

where $\psi(z,s)=(z,w)$ and $s\cdot s'$ is given by (1.27). It can be easily checked that the right-hand side of (1.28) is equal to

$$(1.29) \qquad (z,w_1+s_1',w_2+s_2',w_3+s_3'+2w_1s_2'-2s_1'w_2) = (z,w\cdot s'),$$

where $w\cdot s'$ denotes the extension of the group multiplication (1.27) to $\mathbb{C}^3$, the complexification of G.

The holomorphic extension to $\mathbb{C}^4$ of the action is then given by (1.28) and (1.29) as

$$H((z,w),s') = (z, w\cdot s'). \tag{1.30}$$

In general, the formula for the holomorphic extension of the group action, as given by Theorem 2, is not as simple as in this example.

2. Holomorphic extension of group actions. Proof of Theorem 2

We shall now prove Theorem 2. We assume that (0.6) holds where $\mathscr{E}$ is the Lie algebra given by (0.5) We denote by l'' the dimension of $\mathscr{E}$ as a Lie algebra and by l' the dimension of the fiber $\mathscr{E}_m$, $m\in M$, which by assumption is independent of m. By (0.5) we have $l'' \geqslant l' \geqslant l$. We begin with the following lemma.

(2.1) **Lemma.** *Suppose that E is a finite dimensional Lie subalgebra of real vector fields defined on a smooth manifold M of dimension N for which* $\dim E_m = l'$ *is constant in m. Then around every $m_0\in M$ there exist local coordinates $s_1,\dots,s_{l'}$, $u_1,\dots,u_{N-l'}$ such that any $T\in E$ can be written in the form*

$$T = \sum_{k=1}^{l'} c_k(u,s)\frac{\partial}{\partial s_k} \tag{2.2}$$

with c_k smooth, real analytic in s, uniformly in u.

Proof. Since the vector fields in E are real and form a bundle satisfying the integrability condition, by the Frobenius theorem we can find coordinates s and u vanishing at m_0 so that (2.2) holds with c_k smooth. We shall show that there is a smooth change of coordinates, $(u,s)\mapsto(u,s')$ such that in the new coordinates the coefficients in (2.2) are real analytic in s' uniformly in u. Let $\dim E = l''$ and $T_1,\dots,T_{l''}$ a basis of E with $T_1,\dots,T_{l'}$ linearly independent at m_0 and $T_{l'+1},\dots,T_{l''}$ vanishing at m_0. Put $S_j = T_j$ for $1\leqq j\leqq l'$ and $S_j = T_j + \sum_{h=1}^{l'} d_{jk}(u)T_k$, for $l'+1\leqq j\leqq l''$, where the d_{jk} are the uniquely determined smooth functions such that S_j vanishes identically for $s=0$, for j between $l'+1$ and l''. We introduce the exponential coordinates (u,s') defined by

$$(u,s')\mapsto(u,s) = \left(u, \exp\left(\sum_{j=1}^{l'} s'_j S_j\right)\cdot 0\right)$$

where 0 denotes the origin in $\mathbb{R}^{l'}$. We claim that for any k, $1\leqslant k\leqslant l''$,

$$(\exp tS_k)\left(\exp\sum_{j=1}^{l'} s'_j S_j\right)\cdot 0 = \exp\left(\sum_{p=1}^{l'} b_{kp}(t,u,s')S_p\right)\cdot 0$$

with $b_{kp}(t,u,s')$ real analytic in t and s' small, uniformly in u. Indeed, since the S_j, $1\leqq j\leqq l''$, form a finite dimensional Lie algebra for each u fixed we have, by the Campbell–Hausdorff formula

$$(\exp S_k)\left(\exp\sum_{j=1}^{l'} s'_j S_j\right) = \left(\exp\sum_{p=1}^{l'} b_{kp}(t,u,s')S_p\right)\left(\exp\sum_{p=l'+1}^{l''} b_{kp}(t,u,s')S_p\right).$$

Since the coefficients in the Campbell–Hausdorff formula depend analytically on the structure constants of the Lie algebra, the functions b_{kp} are analytic in t, s' and $d_{jk}(u)$. The vanishing of S_p, $l'+1 \leqq p \leqq l''$ at $s=0$ implies that

$$\left(\exp \sum_{p=l'+1}^{l''} b_{kp}(t,u,s')S_p\right)\cdot 0 = 0,$$

which completes the proof of the claim above. Finally, in the coordinates (u,s') we have $S_k = \sum_{p=1}^{l'} \frac{\partial b_{kp}}{\partial t}(0,u,s')\frac{\partial}{\partial s'_p}$, which completes the proof of the lemma.

(2.3) **Lemma.** *Under the assumptions of Theorem 2 there exist local coordinates* (u,s) *on* M *such that there is a basis* L_j *of* $\mathbb{L}$ *which can be written*

$$L_j = \sum_{p=1}^{2n+l-l'} a_{jp}(u,s)\frac{\partial}{\partial u_p} + \sum_{q=1}^{l'} b_{jq}(u,s)\frac{\partial}{\partial s_q}, \tag{2.4}$$

$j=1,\dots,n$, *where the* a_{jp} *and* b_{jq} *are smooth, and real analytic in* s *uniformly for fixed* u *in a neighborhood of* 0.

Proof. We choose coordinates (u,s) given by Lemma (2.1) and $T_1,\dots,T_{l''}$ a basis of the Lie algebra $\mathscr{E}$ written in the form (2.2). We may assume that $T_1,\dots,T_{l'}$ are linearly independent; therefore we have $T_j = \sum_{k=1}^{l'} c_{jk}(u,s)T_k$, $l'+1 \leqq j \leqq l''$, with c_{jk} smooth and real analytic in s uniformly in u. Since $\dim \mathscr{V}_m = n$ for $m \in M$, by elementary linear algebra arguments we can find a local basis $L_1,\dots,L_n$ of $\mathbb{L}$ of the form

$$L_j = \frac{\partial}{\partial u_j} + \sum_{k=J+1}^{2n+l-l'} a_{jk}(u,s)\frac{\partial}{\partial u_k} + \sum_{k=n-J+1}^{l'} b_{jk}(u,s)T_k, \quad 1 \leqq j \leqq J \tag{2.5}$$

and

$$L_j = \sum_{k=J+1}^{2n+l-l'} a_{jk}(u,s)\frac{\partial}{\partial u_k} + T_{j-J} + \sum_{k=n-J+1}^{l'} b_{jk}(u,s)T_k, \quad J+1 \leqq j \leqq n, \tag{2.6}$$

with $0 \leqq J \leqq n$ and a_{jk}, b_{jk} smooth, $a_{jk}(0)=0$, for $J+1 \leqq j \leqq n$.

We shall prove that in fact both a_{jk} and b_{jk} in (2.5) and (2.6) are real analytic in s, which will yield (2.4). Using (0.8) and (2.5), (2.6) we obtain for j, k, $1 \leqq j \leqq l'$, $1 \leqq k \leqq n$,

$$[T_j, L_k] = \sum_{t>J} \beta_{jkt}(u,s)L_t. \tag{2.7}$$

Since $\mathscr{E}$ is a Lie algebra we obtain, after substituting for T_j, $l'+1 \leqq j \leqq l''$,

$$[T_j, T_q] = \sum_{r=1}^{l'} f_{jqr}(u,s)T_r, \quad 1 \leqq j,\ q \leqq l', \tag{2.8}$$

with f_{jqr} smooth and real analytic in s, uniformly in u. By Lemma (2.1) we may write for $1 \leqq p \leqq l'$, $1 \leqq k \leqq 2n+l-l'$

$$\left[T_p, \frac{\partial}{\partial u_k}\right] = \sum_{r=1}^{l'} \gamma_{pkr}(u,s)T_r \tag{2.8'}$$

with γ_{pkr} real analytic in s, uniformly in u. Calculating the left hand side of (2.7) and making use of (2.8) and (2.8′) yields

$$\sum_{p>J}(T_ja_{kp})\frac{\partial}{\partial u_p}+\sum_{q>n-J}(T_jb_{kq})T_q+\sum_{\substack{q>n-J\\1\leq r\leq l'}}b_{kq}f_{jqr}(s)T_r+\sum_{1\leq r\leq l'}f_{j,k-J,r}T_r$$

(2.9)
$$+\sum_{p>J}\sum_{1\leq r\leq l'}a_{kp}\gamma_{jpr}T_r+\sum_{1\leq r\leq l'}\tilde{\gamma}_{jkr}T_r$$

$$=\sum_{t>J}\beta_{jkt}\left[\sum_p a_{tp}\frac{\partial}{\partial u_p}+\sum_{q>n-J}b_{tq}T_q+T_{t-J}\right],$$

with $\tilde{\gamma}_{jkr}=\begin{cases}\gamma_{jkr}, & 1\leq k\leq J\\ 0 & k>J\end{cases}$ and the convention that $f_{j,k-J,r}=0$ for $k\leq J$.

Taking the coefficients of $T_1,\ldots,T_{n-J}$ in (2.9) we obtain

(2.10) $$\tilde{\gamma}_{jkr}+\sum_{p>J}a_{kp}\gamma_{jpr}+\sum_{q>n-J}b_{kq}f_{jqr}+f_{j,k-J,r}=\beta_{j,k,J+r},\quad r=1,\ldots,n-J.$$

Similarly, taking the coefficients of $T_{n-J+1},\ldots,T_{l'}$ in (2.9) yields

(2.11) $$\tilde{\gamma}_{jkq}+\sum_{p>J}a_{kp}\gamma_{jpq}+T_jb_{kq}+\sum_{r>n-J}b_{kr}f_{jrq}+f_{j,k-J,q}=\sum_{t>J}\beta_{jkt}b_{tq},$$

for all j, k, q, $1\leq j\leq l'$, $n-J<q\leq l'$, $1\leq k\leq n$. Also, taking the coefficient of $\partial/\partial u_p$ in (2.9) we obtain

(2.12) $$T_ja_{kp}=\sum_{t>J}\beta_{jkt}a_{tp}.$$

We denote by A the vector (a_{kp}), by B the vector (b_{kq}) and put $C=\begin{pmatrix}A\\B\end{pmatrix}$. After substituting (2.10) in (2.11) and (2.12) we obtain

(2.13) $$T_jC=F_j(s,u,C),\quad j=1,\ldots,l'$$

where F is real analytic in s, smooth in u, and quadratic in the components of C. The analyticity of C with respect to s, uniformly in u, follows from the ellipticity of the system (2.13).

The proof of Lemma (2.3) is now complete.

(2.14) **Lemma.** *Let M be a* CR *manifold of dimension $2n+l$, and codimension l. Assume that around $m_o\in M$ there are coordinates $u_1,\ldots,u_{2n+l-l'}$, $s_1,\ldots,s_{l'}$ and a basis $L_j, j=1,\ldots,n$, of $\mathbb{L}$, of the form (2.4) with coefficients real analytic in s uniformly in u, and such that L_j, $\bar{L}_j$, $\partial/\partial s_k$, $1\leq j\leq n$, $1\leq k\leq l'$, span the tangent space to M at m_0. Then there is a local embedding of M into $\mathbb{C}^{n+l}$ given by a* CR *mapping $(\psi_1(u,s),\ldots,\psi_{n+l}(u,s))$ with ψ_j real analytic in s. In addition, all such embeddings are equivalent up to biholomorphism.*

Proof. First we introduce new real variables $s'_1,\ldots,s'_{l'}$ and denote by W the subbundle of $\mathbb{C}T(M \times U)$, $U \subset \mathbb{R}^{l'}$ open, whose sections are spanned by Λ_j, $1 \leqq j \leqq n$, and $(\partial/\partial s_k) + i(\partial/\partial s'_k)$, $1 \leqq k \leqq l'$, where Λ_j is obtained from L_j by replacing s by $s + is'$ in the coefficients. It follows from the assumptions of the lemma that, after shrinking M and U

$$[W, W] \subset W, \quad W + \bar{W} = \mathbb{C}T(M \times U).$$

The embeddability then follows from the Newlander–Nirenberg Theorem [8] (see also Nirenberg [9].) In fact, there is a smooth change of coordinates $x = x(u, s)$, $y = y(u, s), t = t(u, s)$ with x, $y \in \mathbb{R}^{n+l}$, $t \in \mathbb{R}^{l'-l}$, such that $\mathscr{W}$ is spanned by $\frac{\partial}{\partial \bar{z}_j}$, $1 \leqq j \leqq n + l$, $\frac{\partial}{\partial t_k}$, $1 \leqq k \leqq l' - l$.

To prove the last statement of the lemma, we observe that any CR function f on M, real analytic with respect to s extends to a function annihilated by the sections of $\mathscr{W}$, i.e. by Λ_j and $\frac{\partial}{\partial s_k} + i\frac{\partial}{\partial s'_k}$. Therefore, $f(u, s) = H(Z_1(u, s), \ldots, Z_{n+l}(u, s))$ with $Z_j(u, s) = x_j(u, s) + iy_j(u, s)$, with H holomorphic in $\mathbb{C}^{n+l}$. This proves Lemma (2.14).

By Lemma (2.1) we can find coordinates (u, s) on M for which the group acts on s alone i.e. for $g \in G$

$$H((u, s), g) = (u, a(u, s, g)), \tag{2.15}$$

where $a(u, s, g)$ is a local group action on $\mathbb{R}^{l'}_s$ for u fixed. It follows from (1.4), (1.5) and Lemma (2.1) that $a(u, s, g)$ is smooth and real analytic in s and g uniformly in u.

(2.16) **Lemma.** *Under the assumptions of Theorem 2, if h is a* CR *function defined on M and $g \in G$, then the function $(u, s) \mapsto h(u, a(u, s, g))$ is again* CR, *where (u, s) are coordinates given by Lemma* (2.1); *we have used the notation* (2.15).

Proof. Let $L_1, \ldots, L_n$ be a basis of $\mathbb{L}$ and let $f_t(m)$ be the flow of a vector field $T \in \mathscr{E}$, with $f_0(m) = m$. By (0.8), as in the proof of Theorem 1, we obtain (1.8) with ψ replaced by h. The Lemma follows since $L_j h = 0$.

End of proof of Theorem 2.

Let (u, s) be the coordinates on M given by Lemma (2.1) and $\psi(u, s)$ the embedding of M in $\mathbb{C}^{n+l}$ given by Lemma (2.4). Using Lemma (2.16) we see that for $g \in G$ near the identity the mapping

$$(u, s) \mapsto \psi(u, a(u, s, g))$$

defines another embedding of M in $\mathbb{C}^{n+l}$. By the uniqueness of Lemma (2.14) we conclude that there exists a biholomorphic mapping $Z \to H_g(Z)$ in $\mathbb{C}^{n+l}$ such that

$$\psi(u, a(u, s, g)) = H_g(\psi(u, s)), \tag{2.17}$$

which proves (0.9).

To complete the proof of Theorem 2 it will suffice to show that if

$$\psi' : (u, s) \mapsto \psi'(u, s)$$

is any CR embedding of M into $\mathbb{C}^{n+l}$ for which the group action extends holomorphically then ψ' is real analytic as a function of s. (For then, the previous argument shows that ψ' is biholomorphically equivalent to the one constructed above.) Since the mapping $(u,g)\mapsto(u,a(u,s,g))$ is real analytic and surjective, it suffices to show that $g\mapsto\psi'(u,a(u,s,g))$ is real analytic uniformly in u. Since

(2.18) $$H(\psi'(u,s),g)=\psi'(u,a(u,s,g))$$

we have

$$H(\psi'(u,0),g)=\psi'(u,a(u,0,g)).$$

Hence it suffices to show that $g\mapsto H(\psi'(u,0),g)$ is real analytic. Since by assumption $Z\to H(Z,g)$ is holomorphic the proof of the theorem will be completed by the following result.

(2.18) **Lemma.** *Let $H:U\times V\to\mathbb{C}^n$ be a smooth local group action, where $U\subset\mathbb{C}^n$ is a neighborhood of* 0 *and V is a neighborhood of the identity in a Lie group G. Suppose that $Z\mapsto H(Z,g)$ is holomorphic for each $g\in V$. Then $g\mapsto H(Z,g)$ is real analytic in a neighborhood of the identity* (*for Z fixed*).

Proof. Let $(t_1,\ldots,t_p)$ be a system of exponential coordinates near the identity in G. Then by the Baker–Campbell–Hausdorff formula the product $(t_1,\ldots,t_p)\cdot(t'_1,\ldots,t'_p)=m(t,t')$ is of the form

(2.19) $$m_k(t,t')=t_k+t'_k+O(|t||t'|).$$

By the group law

(2.20) $$H(Z,(t_1,\ldots,t_p)\cdot(t'_1,\ldots,t'_p))=H(H(Z,(t_1,\ldots,t_p)),(t'_1,\ldots,t'_p)).$$

Differentiating (2.20) with respect to t'_j and setting $t'_j=0$ yields

(2.21) $$\Sigma H_{t_k}(Z,t)\frac{\partial m_k}{\partial t'_j}(t,0)=H_{t_j}(H(Z,t),0).$$

The functions $m_k(t,t')$ are real analytic and by using (2.19) the matrix $\dfrac{\partial m_k}{\partial t'_j}(t,0)$ is invertible near $t=0$. Since $H_{t_j}(Z,0)$ is holomorphic in Z we obtain that $H(Z,t)$ satisfies an elliptic system, for fixed Z, of the form

$$v_{t_k}=A_k(v,t),\quad k=1,\ldots,p,$$

with A_k real analytic. Hence $H(Z,t)$ is real analytic in t, which proves the lemma.

(2.22) *Examples.* We give here two explicit examples of CR manifolds with embeddings for which their transversal group action does not extend holomorphically. For the simplest example, we let

$$M=\{(z,\omega)\in\mathbb{C}^2:\operatorname{Im}w=f(\operatorname{Re}w)\},$$

where f is a real valued, smooth function, $f(0)=0$, which is not real analytic near 0. Then the mapping

$$H:(z,s+if(s))\mapsto(z,s+t+if(s+t))$$

defines a global 1-parameter CR group action on M. Since M may also be embedded in $\mathbb{C}^2$ with image $\{(z,w)\in\mathbb{C}^2 : \operatorname{Im} w = 0\}$, the uniqueness of Theorem 2 proves that H does not extend holomorphically. (This could also be proved directly by using Lemma (2.18).)

We can also construct such an example for an embedded strictly pseudoconvex manifold. Take

$$M = \{(z,w)\in\mathbb{C}^2 : w = s + i|z|^2 + f(z,\bar{z},s)\},$$

where $s\in\mathbb{R}$, f a non real analytic solution of $\left(\frac{\partial}{\partial\bar{z}} - iz\frac{\partial}{\partial s}\right)f = 0$, $f(0)=0$. The mapping

$$(z, s + i|z|^2 + f(z,\bar{z},s)) \mapsto (z, s + t + i|z|^2 + f(z,\bar{z},s+t))$$

defines a CR 1-parameter group action which does not extend holomorphically for this embedding.

3. FBI transform on a group. Proof of Theorem 3

We shall prove Theorem 3 by defining an analogue of the FBI (Fourier–Bros–Iagolnitzer) transform (see [11]) on a Lie group, and then using an argument similar to that used in [6], where the case of an abelian group is discussed. (See also [2] where a related transformation is discussed.)

Let G be a Lie group of dimension l and $X_1,\dots,X_l$ be a basic of $\mathfrak{g}$. If V is a small neighborhood of the identity in G we denote by Φ the real analytic diffeomorphism given by

$$\Phi : V \to \mathbb{R}^l, \quad \Phi(a) = \alpha = (\alpha_1,\dots,\alpha_l), \tag{3.1}$$

where $a = \exp(\Sigma\alpha_j X_j)$. We denote by da the left Haar measure on G. For a function u defined in V with compact support we have

$$\int_G u(a)\,da = \int_{\mathbb{R}^l} u(\Phi^{-1}(\alpha))\sigma(\alpha)\,d\alpha, \tag{3.2}$$

where $\sigma(\alpha)$ is a real analytic function. We normalize the Haar measure by taking $\sigma(0)=1$. If f and h are in $L^1(G,da)$ we write

$$f * h(a) = \int_G f(b^{-1}a)h(b)\,db. \tag{3.3}$$

If S is a right invariant vector field on G we have

$$S(f*h)(a) = (f*Sh)(a). \tag{3.4}$$

For $h\in C_0^\infty(V)$ we define the FBI transform $F(h,a,\xi)$, $\xi\in\mathbb{R}^l$, $a\in G$ near the identity by

$$F(h,a,\xi) = \int_G \exp(i\Phi(b^{-1}a)\xi - |\xi|[\Phi(b^{-1}a)]^2)\Delta(\Phi(b^{-1}a),\xi)h(b)\,db, \tag{3.5}$$

where $\Delta(\alpha,\xi) = \left(1 + i\sum_{j=1}^{l}\alpha_j\frac{\xi_j}{|\xi|}\right)$ with $\alpha=(\alpha_1,\dots,\alpha_l)$. A standard integration by parts argument shows that $F(h,a,\xi)$ is rapidly decreasing in ξ.

(3.6) **Lemma.** *The inversion formula*

$$h(a)=\frac{1}{(2\pi)^l}\int_{\xi\in\mathbb{R}^l} F(h,a,\xi)d\xi$$

holds for $h\in C_0^\infty(V)$ and $a\in G$, a near the identity.

Proof. We note first that

$$h(a)=\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^l}\int\int_{G\times\mathbb{R}^l_\xi}\exp(i\Phi(b^{-1}a)\xi-\varepsilon^2|\xi|^2)h(b)dbd\xi.$$

Indeed, this is a slight variation of Lemma 1 of [2]. By making the change of contour $\xi\mapsto\xi+i\Phi(b^{-1}a)\frac{\xi}{|\xi|}=\theta$ as in the Euclidean case (see e.g. [6, 11]), we obtain

$$h(a)=\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^l}\iint_{\xi\in\mathbb{R}^l b\in G}\exp(i\Phi(b^{-1}a)\xi-|\xi|(\Phi(b^{-1}a))^2-\varepsilon^2[\theta]^2)\Delta(\Phi(b^{-1}a),\xi)h(b)dbd\xi,$$

which proves the lemma by a standard limiting argument.

Now let M satisfy the assumptions of Theorem 3 and (u,s) be coordinates as introduced in the proof of Theorem 2. Here $u\in\mathbb{R}^{2n}$ and $s\in\mathbb{R}^l$, where $\mathbb{R}^l$ is locally identified with the group G. We consider the embedding given in the proof of Theorem 2. By choosing $S_1,\dots,S_l$ to be right invariant vector fields on G as in [6] we may write a basis of $\mathbb{L}$ in the form

$$L_j=\frac{\partial}{\partial\bar z_j}+\sum_{p=1}^{l}a_j^p(z,\bar z)S_p. \tag{3.7}$$

Now let h be a CR function on M. We shall introduce a mini-FBI transform of h adapted to the group structure. We parametrize M by (x,y,s), and consider the embedding defined by $z=x+iy\in\mathbb{C}^n$, $w=\psi(z,\bar z,s)\in\mathbb{C}^l$ with ψ satisfying $\psi(0)=0$ and $\det\psi_s(0)\neq 0$ and ψ analytic in s. By the implicit function theorem we can find $G(z,\bar z,w)$, analytic in w, such that $\psi(z,\bar z,G(z,\bar z,w))\equiv w$. Let $\chi\in C_0^\infty(V)$ with $\chi\equiv 1$ near the identity. We write

$$F(\chi h,z,\bar z,w,\xi)=\int_G\exp(i\Phi(s^{-1}G(z,\bar z,w))\xi-|\xi|(\Phi(s^{-1}G(z,\bar z,w)))^2)\cdot\Delta(\Phi(s^{-1}G(z,\bar z,w),\xi)\chi(s)h(z,\bar z,s)ds. \tag{3.8}$$

This is defined for (x,y) small and for $w\in\mathbb{C}^l$ small; the group multiplication has been extended to the complexification of G. Note that if $w=\psi(z,\bar z,s)$ then (3.8) is the FBI transform, as defined by (3.5), for the function $\chi(\cdot)h(z,\bar z,\cdot)$. Therefore, it follows from Lemma (3.6) that we have for $(z,\bar z,s)$ small

$$h(z,\bar z,s)=\frac{1}{(2\pi)^l}\int_{\mathbb{R}^l}F(\chi h,z,\bar z,\psi(z,\bar z,s),\xi)d\xi \tag{3.9}$$

We now modify the argument for the proof of Theorem II.3 of [6] for the present situation. Let $\Gamma_1,\dots,\Gamma_r$ be strictly convex cones contained in $\mathbb{R}^l$ with

$$\mathbb{R}^l=\bigcup_{j=1}^r\Gamma_j\ \text{meas.}(\Gamma_j\cap\Gamma_k)=0,\quad j\neq k.$$

For a CR function h we define h_j by

$$h_j(z,\bar{z},s)=\frac{1}{(2\pi)^l}\int_{\Gamma_j} F(\chi h,z,\bar{z},\psi(z,\bar{z},s),\xi)d\xi. \tag{3.10}$$

It follows from (3.9) that $h=\sum_{j=1}^{r} h_j$. We define $H_j(z,\bar{z},w)$ by

$$H_j(z,\bar{z},w)=\frac{1}{(2\pi)^l}\int_{\Gamma_j} F(\chi h,z,\bar{z},w,\xi)d\xi \tag{3.11}$$

for $(z,w)\in\mathscr{W}_j$, where $\mathscr{W}_j$ is a wedge given by $\{\operatorname{Im} G(z,\bar{z},w)\in\mathscr{C}_j\}$, with z,w small and $\mathscr{C}_j\Subset\Gamma_j^0$, where Γ_j^0 is the polar of Γ_j. To show that the right hand side of (3.11) is defined and holomorphic as a function of w, it suffices to show

$$|\exp(i\Phi(s^{-1}(s'+it))\xi-|\xi|[\Phi(s^{-1}(s'+it))]^2)|\leqq\exp(-\alpha|\xi||t|) \tag{3.12}$$

for some $\alpha>0$, $\xi\in\Gamma_j$ and $s\in\operatorname{supp}\chi$, $t\in\mathscr{C}_j$, s' and t sufficiently small. We note that

$$\Phi(s^{-1}(s'+it))=s'-s+it+O(|s'||s|,|s||t|) \tag{3.13}$$

(by the Campbell–Hausdorff formula). Clearly it suffices to prove (3.12) with $s'=0$. It is then an easy consequence of (3.13).

(3.14) **Lemma.** *For every $k=1,\dots,n$, $j=1,\dots,r$, there exists $K_{jk}\in C^\infty(\mathscr{O},\mathscr{H}(U))$, where $\mathscr{O}$ is an open neighborhood of 0 in $\mathbb{R}^{2n}$ and U an open neighborhood of 0 in $\mathbb{C}^l$ (here $C^\infty(\mathscr{O},\mathscr{H}(U))$ denotes smooth functions on $\mathscr{O}$ with values in $\mathscr{H}(\mathscr{U})$, holomorphic functions in U, such that $L_k h_j(z,\bar{z},s)=K_{jk}|_M$ for z, s small.*

Proof. By (3.4) and (3.7) we obtain

$$L_k h_j(z,\bar{z},s)=\iint_{s'\in G\,\xi\in\Gamma_j}\exp(i\Phi(s'^{-1}s)\xi-|\xi|[\Phi(s'^{-1}s)]^2)(L_k\chi)(z,\bar{z},s')\cdot\Delta(\Phi(s'^{-1}s),\xi)h(z,\bar{z},s')ds'd\xi. \tag{3.15}$$

Since $L_k\chi\equiv 0$ near the origin we can replace s by $G(z,\bar{z},w)$ in the right hand side of (3.15) to obtain the desired function K_{jk}, which proves the lemma.

Next we claim that

$$\partial_{\bar{z}_k}H_j(z,\bar{z},w)=K_{jk}(z,\bar{z},w) \tag{3.16}$$

for $(z,w)\in\mathscr{W}_j$. Indeed, we have, since $L_k\psi\equiv 0$ and using Lemma (3.14),

$$\partial_{\bar{z}_k}H_j(z,\bar{z},w)|_{w=\psi(z,\bar{z},s)}=L_k(H_j(z,\bar{z},\psi(z,\bar{z},s))=K_{jk}(z,\bar{z},\psi(z,\bar{z},s)). \tag{3.17}$$

Hence the functions $\partial_{\bar{z}_k}H_j(z,\bar{z},w)$ and $K_{jk}(z,\bar{z},w)$ agree on the hypersurface $w=\psi(z,\bar{z},s)$ and are both holomorphic in $\mathscr{W}_j$. By uniqueness, (3.16) holds.

Using (3.16) we can find $R_j\in C^\infty(\mathscr{O},\mathscr{H}(U))$ (after shrinking $\mathscr{O}$ and U) so that $\partial_{\bar{z}_k}R_j(z,\bar{z},w)=\partial_{\bar{z}_k}H_j(z,\bar{z},w)$. Now if we take

$$K_j=H_j-R_j+\frac{1}{r}\sum_{p=1}^{r}R_p$$

then K_j is holomorphic in $\mathscr{W}_j$ and $h=\Sigma bK_j$.

(3.18) *Remark.* Theorem 3 is a generalization of Theorem II.3 of [6], where the result is proved for the special case of an abelian group G. The result is false for general embedded CR manifolds, as was first shown by an example of Trépreau [13] (see also [5].) Also note that the mini-FBI transform defined by (3.5) and (3.8) differs, in the non-abelian case, with the one used in [3] for general generic submanifolds in $\mathbb{C}^{n+l}$.

References

1. Baouendi, M.S., Chang, C.H., Treves, F.: Microlocal hypo-analyticity and extension of CR functions. J. Differ. Geom. **18**, 331–391 (1983)
2. Baouendi, M.S., Rothschild, L.P.: Analytic approximation for homogeneous solutions of invariant differential operators on Lie groups. Asterisque **131**, 189–199 (1985)
3. Baouendi, M.S., Rothschild, L.P.: Normal forms for generic manifolds and holomorphic extension of CR functions. J. Differ. Geom. **25**, 431–467 (1987)
4. Baouendi, M.S., Rothschild, L.P.: Embeddability of abstract CR structures and integrability of related systems. Ann. Inst. Fourier **37**, 131–141 (1987)
5. Baouendi, M.S., Rothschild, L.P.: Extension of holomorphic functions in generic wedges and their wave front sets. Commun. Part. Differ. Equations **13**, 1441–1466 (1988)
6. Baouendi, M.S., Rothschild, L.P., Treves, F.: CR structures with group action and extendability of CR functions. Invent. Math. **82**, 359–396 (1985)
7. Jacobowitz, H.: The canonical bundle and realizable CR hypersurfaces. (preprint)
8. Newlander, A., Nirenberg, L.: Complex coordinates in almost complex manifolds. Ann. Math. **65**, 391–404 (1957)
9. Nirenberg, L.: A Complex Frobenius Theorem. Seminar on analytic functions I, pp. 172–189. Princeton (1957)
10. Nomizu, K.: Lie groups and differential geometry. Math. Soc. Japan (1956)
11. Sjöstrand, J.: Singularités analytiques microlocales. Astérisque **95**, 1–166 (1982)
12. Tanaka, N.: On the pseudoconformal geometry of hypersurfaces of the space of n complex variables. J. Math. Soc. Japan **14**, 397–429 (1962)
13. Trepreau, J.-M.: Une fonction CR non décomposable en somme de valeurs au bord de fonctions holomorphes, (unpublished)
14. Varadarajan, V.S.: Lie groups, Lie algebras, and their representations. Englewood Cliffs, NJ: Prentice-Hall 1974

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