UNBOUNDED CONJUGACY CLASSES IN LIE GROUPS AND LOCATION OF CENTRAL MEASURES

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1. Introduction

For a locally compact group G, Tits [14] has described the subgroup $B(G)$ of all elements in G which have precompact conjugacy classes. To use this result for analysis on G it is important to have information about conjugacy classes of whole neighborhoods in G, as well as those of single points. In particular, it is natural to ask whether an arbitrary $g \in G \sim B(G)$ has a neighborhood U with infinitely many disjoint conjugates $\alpha_g(U) = gUg^{-1}$, $g \in G$. Although this is true for semisimple connected Lie groups [10], we show that it fails to hold in general. Nevertheless, the unbounded conjugacy classes in G do possess certain uniformity properties. Using the structure theory of Lie groups, the authors describe the uniformity properties of the unbounded conjugacy classes in any connected locally compact group. These results are then applied directly to prove that the support of any finite central measure on G must be contained in $B(G)$. Locating supports in this way greatly simplifies the harmonic analysis of such measures. Finally the authors refine Tits' description of $B(G)$, so that these results can be applied to a variety of groups.

1.1 *Definition*. Let X be a locally compact space, and $G \times X \to X$ a jointly continuous action, and $A \subset X$ a closed G-invariant set. A *layering* of X terminating with A is any sequence $X = X_m \supset X_{m-1} \supset \ldots \supset X_0 = A$ of closed G-invariant sets such that each point x in the kth "layer" $X_k \sim X_{k-1}$ has a relative neighborhood in $X_k \sim X_{k-1}$ with infinitely many disjoint G-transforms. If $X = A$ the conditions are vacuously satisfied.

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Let $\mathfrak{J}(G)$ be the group of all inner automorphisms α_g of G. Our main result gives the existence of a layering of G under conjugation $\mathfrak{J}(G) \times G \to G$. The proof is first reduced to the case of a connected Lie group without proper compact normal subgroups. Lie theory is then used to produce a layering that terminates with the centralizer $Z_G(N)$ of the (connected) nilradical N. Applying the (known) result for the semisimple case, one then extends this layering so that it terminates in a certain closed characteristic subgroup A whose identity component is a vector group; in fact, it is the center of the nilradical. Finally, one is reduced to studying the affine action of a one parameter group, or of a connected semisimple Lie group, on a finite dimensional real vector space. This reduces to questions about *linear* actions, which are analyzed by elementary methods. Our principal result along these lines is the following.

1.2 THEOREM. *Let G be a locally compact group and $G \times V \to V$ an affine action on a real finite dimensional vector space. Let V_c be the elements in V with bounded G-orbits. Then V_c is a G-invariant affine variety (possibly empty) and there is a layering $V = V_m \supset \ldots \supset V_0 = V_c$ consisting of G-invariant affine varieties.*

This result seems to be of independent interest even when $G = \mathbf{R}$, because of its relationship to dynamical systems.

For a locally compact group G, let $\mathcal{A}(G)$ denote the group of all bicontinuous automorphisms of G, and $\mathfrak{J}(G)$ the subgroup of inner automorphisms. If $x \in G$, then O_x denotes the conjugacy class—its $\mathfrak{J}(G)$-orbit

1.3 *Definition.* For $x \in G$ we say that the class O_x is (i) *bounded* if O_x has compact closure, (ii) *unbounded* if O_x has noncompact closure, (iii) *uniformly unbounded* if there exists a neighborhood U of x with infinitely many pairwise disjoint conjugates.

The set $B(G) = \{x \in G\colon O_x \text{ is bounded}\}$ is a normal (in fact, characteristic) subgroup in G. Tits [14, p. 38] has shown that $B(G)$ is closed in G if G is a connected group; this means that $B(G)$ is an $[FC]^-$ group, in the sense of [3]. In section 4 we give an example of a (5-dimensional) nilpotent group and elements $x \in G \sim B(G)$ such that no neighborhood of such a point has infinitely many disjoint conjugates, even though the class O_x is unbounded.

Let A be a closed $\mathfrak{J}(G)$-invariant set in G, and $G = X_m \supset \ldots \supset X_0 = A$ a layering terminating with A. Points off A must have unbounded conjugacy classes, so that $A \supset B(G)$. Points in the first layer $X \sim X_{m-1}$ actually have uniformly unbounded conjugacy classes, but for $x \in X_{m-1}$ (usually a lower dimensional variety) we must restrict attention to *relative* neighborhoods, see section 4.

Here is our main result on unboundedness of conjugacy classes.

1.4 THEOREM. *If G is any connected locally compact group, then there exists a layering of G that terminates with the closed subgroup $B(G)$; that is, there are closed $\mathcal{J}(G)$-invariant subsets $G=G_m\supset\ldots\supset G_0=B(G)$ such that every point $x\in G_k\sim G_{k-1}$ has a relative neighborhood in G_k with infinitely many disjoint conjugates.*

Clearly a layering cannot terminate with a set smaller than $B(G)$.

If $x\notin B(G)$ the uniform unboundedness properties of its conjugacy class can be used to draw immediate conclusions about the location of supports of central measures on G. Let $C_c(G)$ be the continuous complex values functions with compact support on G, equipped with the inductive limit topology. A Radon measure $\mu\in C_c(G)^*$ is *invariant* under $\mathcal{J}(G)$ (or, in some accounts, a *central measure*) if

$$\langle\mu, f\rangle=\langle\mu, f\circ\alpha_x\rangle \text{ all } \alpha_x\in\mathcal{J}(G),\quad f\in C_c(G),$$

where $(f\circ\alpha_x)(g)=f(xgx^{-1})$. Now $M(G)$, the measures with finite total variation, is a Banach *-algebra under convolution and is the dual of the Banach space $C_0(G)$ of continuous functions which vanish at infinity. Letting δ_x be the point mass at $x\in G$, it is easily seen that $\mu\in M(G)$ is invariant $\Leftrightarrow\delta_x*\mu*\delta_{x^{-1}}=\mu$ for all $x\in G\Leftrightarrow\nu*\mu=\mu*\nu$ for all $\nu\in M(G)\Leftrightarrow\mu$ is in the center of the Banach algebra $M(G)\Leftrightarrow\mu(\alpha_x(E))=\mu(E)$ for all Borel sets $E\subset G$ and all $x\in G$. The Radon-Nikodym theorem shows that the absolute value $|\mu|$ is invariant if μ is invariant; since supp $(\mu)=$ supp $(|\mu|)$, all questions concerning supports can be decided by examining only non-negative central measures.

If $x\in G$ has a uniformly unbounded $\mathcal{J}(G)$-orbit, then x cannot be in supp (μ) for any positive central measure $\mu\in M(G)$; for any neighborhood U of x we get $\mu(U)>0$, and if U has infinitely many disjoint conjugates $\{\alpha_i(U): i=1, 2, \ldots\}$, then $\mu(\alpha_i(U))=\mu(U)$ and $\mu(G)\geqslant\sum_{i=1}^{\infty}\mu(\alpha_i(U))=+\infty$. If we are given a finite positive central measure and a layering $G=X_m\supset\ldots\supset X_0=A$, then by examining orbits in $X_m\sim X_{m-1}$ we conclude that supp $(\mu)\subset X_{m-1}$. But now μ may be regarded as a finite $\mathcal{J}(G)$-invariant measure on the locally compact space X_{m-1}. In discussing supports it is only necessary to examine relative neighborhoods within X_{m-1}. Since orbits of points in $X_{m-1}\sim X_{m-2}$ are uniformly unbounded with respect to X_{m-1}, we conclude that supp $(\mu)\subset X_{m-2}$. By induction, we conclude that supp $(\mu)\subset A$. Applying Theorem 1.4 we get:

1.5 THEOREM. *All finite central measures on a connected locally compact group G are supported on the closed subgroup $B(G)$.*

Now $B(G)$ always has a simple structure, see section 3, and in many interesting cases reduces to the center of G, see section 9. Section 9 is devoted to a refinement of Tits' description of $B(G)$ in important special cases.

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In dealing with connected Lie groups we shall refer to the following closed subgroups: (i) $R = \mathrm{rad}\,(G)$, the radical; (ii) N = nilradical; (iii) $Z(N)$ = center of the nilradical; (iv) $Z_G(N)$ = centralizer of N in G; (v) $Z(G)$ = center of G; (vi) $K(G)$ = the maximal compact normal subgroup in G. The identity component of a group H is indicated by H_0. For the existence of $K(G)$ in connected locally compact groups, see [5; p. 541]. We will also write $[x, y] = xyx^{-1}y^{-1} = \alpha_x(y)\,y^{-1}$ for the commutator of two group elements, and $[A, B] = \{[a, b]: a \in A, b \in B\}$ for subsets A, B of G.

2. Basic combinatorial results on layerings

Here we set forth simple facts about layerings which will be used throughout our discussion. In particular, they allow us to reduce the proof of Theorem 1.4 to the case of a connected Lie group. The first lemma allows us to lift a layering in a quotient group back to a layering of the original group.

2.1 LEMMA. *Let X, Y be two G-spaces, $\pi: X \to Y$ a continuous equivariant map. If $x \in X$ and $G \cdot \pi(x)$ is uniformly unbounded in Y, so is $G \cdot x$ in X. If $Y = Y_m \supset \ldots \supset Y_0 = A$ is a layering in Y, the sets $X_k = \pi^{-1}(Y_k)$ give a layering in X that terminates at $A' = \pi^{-1}(A)$.*

The proof is obvious by lifting disjoint neighborhoods in Y back to X. If H is a closed normal subgroup of G and $\pi: G \to G/H = G'$ is the quotient map, each inner automorphism α_x on G induces an inner automorphism $\beta_x(yH) = \alpha_x(y)H = \pi(\alpha_x(y)) = \alpha_{\pi(x)}(\pi(y))$ on G'. This correspondence maps $\mathfrak{J}(G)$ *onto* $\mathfrak{J}(G')$. The map $\pi: G \to G'$ is equivariant between these actions of G on G and G' respectively. By Lemma 2.1 every layering $G' = X'_m \supset \ldots \supset X'_0 = A'$ in G' lifts back to a layering $X_k = \pi^{-1}(X'_k)$ of G which terminates at $A = \pi^{-1}(A')$.

2.2 LEMMA. *Suppose that A, B are closed $\mathfrak{J}(G)$-invariant sets in G. If there are layerings $G = X_m \supset \ldots \supset X_0 = A$ and $G = Y_n \supset \ldots \supset Y_0 = B$, then there exists a layering of G that terminates with $A \cap B$.*

Proof. The sets $Y'_k = Y_k \cap A$ are closed, $\mathfrak{J}(G)$-*invariant*; we assert that $G = X_m \supset \ldots \supset X_0 = A = Y'_n \supset \ldots \supset Y'_0 = A \cap B$ is a layering. It is only necessary to examine orbits of points $x \in Y'_k \sim Y'_{k-1}$. By hypothesis, there is a relative neighborhood U in Y_k which has infinitely many disjoint conjugates $\alpha_i(U)$. Now $V = U \cap A$ is a relative neighborhood in Y'_k, and since $\alpha_i(V) \subset \alpha_i(U)$ these conjugates are pairwise disjoint within Y'_k. Q.E.D.

If the maximal compact normal subgroup $K(G)$ is factored out of a connected locally compact group G, then the quotient group $G/K(G)$ contains no nontrivial compact normal subgroups. However, the group G may be approximated by Lie groups by factoring out small compact normal subgroups $K_\beta \subset G$ (Yamabe's theorem, see [7, Ch. 4]); since the K_β all lie within $K(G)$, $G/K(G)$ must be a Lie group. For any locally compact group G, and any compact normal subgroup K, $B(G)$ is the inverse image of $B(G/K)$ under the quotient map $\pi: G \to G/K$.

In view of Lemma 2.1, we may pass from G to $G/K(G)$ in proving Theorem 1.4; that is, we are reduced to considering only connected Lie groups without proper compact normal subgroups.

3. Structure of $B(G)$

3.1 LEMMA. *If G is a connected Lie group and if $K(G)_0$ is trivial, then* (i) *its nilradical N is simply connected and* (ii) *$Z(N)$ is a vector group.*

Proof. Property (ii) follows from (i). Obviously $K(N)_0$, being characteristic in N, is trivial if $K(G)_0$ is trivial. Let $\pi: \tilde{N} \to N$ be a universal covering. Then $Z(\tilde{N}) = V$ is connected, hence a vector group. Let W be the vector subspace of V spanned by Ker (π). Then $K = \pi(W) \cong W/\mathrm{Ker}\,(\pi)$ is compact, central in N, and so must be trivial. Thus π is faithful, as required. Q.E.D.

If $K(G)_0$ is trivial G acts via $\mathcal{J}(G)$ as additive (hence $\mathbf{R}$-linear) transformations in $V = Z(N)$, giving us a linear action $G \times V \to V$. Let V_c be the set of elements $v \in V$ with precompact G-orbits; V_c is a G-invariant linear subspace. Tits' elegant analysis [14] of the bounded orbits in G yields the following description of $B(G)$.

3.2 THEOREM (Tits). *Let G be a connected Lie group. If $K(G)_0$ is trivial, then $B(G) = Z(G) \cdot V_c$. Furthermore, $B(G)$ is a closed, characteristic subgroup of G whose connected component is $B(G)_0 = B(G) \cap N = V_c$. If $K_0 = K(G)_0 \neq \{e\}$ then $B(G)$ is the inverse image of $B(G/K_0)$ under $\pi: G \to G/K_0$.*

Tits proves that $B(G) = Z(G)$ for simply connected nilpotent Lie groups. In section 9 we shall calculate $B(G)$ in a number of other cases, thus strengthening the conclusions in [14] in those cases. For example, if G is simply connected solvable and is either complex analytic, real algebraic, or of type (E), then $B(G) = Z(G)$.

4. A counterexample

The following example shows that orbits of points $x \notin B(G)$ can fail to be uniformly unbounded, even though unbounded. Thus the introduction of layerings seems unavoidable. We start by examining a simpler situation, which will recur later on. Let $V=\mathbf{R}^4$ and let $\eta(t)=\operatorname{Exp}(tA)$, $t\in\mathbf{R}$, be a continuous one-parameter subgroup of $GL(V)$ where

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & 0 & 1 & 0 \\ \vdots & & 0 & 1 \\ 0 & \dots & & 0 \end{bmatrix}, \text{ so that } \eta(t)=\begin{bmatrix} 1 & t & t^2/2! & t^3/3! \\ 0 & 1 & t & t^2/2! \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1}$$

This gives a linear action $\mathbf{R}\times V\to V$, with which we may form the semi-direct product group $G=\mathbf{R}\times_\eta V$.

4.1 Example. *If $V_c=\{v\in V$: orbit of v is precompact$\}$, then $V_c=\operatorname{Ker} A$; thus points in $V\sim V_c$ have unbounded orbits. There exist points in $V\sim V_c$ (in fact in* Ker A^2) *such that no neighborhood has infinitely many disjoint transforms under* $\eta(\mathbf{R})$.

Proof. Using the same basis as in (1), we express vectors as column vectors $v=(a_1, a_2, a_3, a_4)$; then

$$\eta(t)\,(v)=\left(a_1+a_2t+a_3\frac{t^2}{2!}+a_4\frac{t^3}{3!},\ a_2+a_3t+a_4\frac{t^2}{2.},\ a_3+a_4t, a_4\right).$$

The polynomials involved are unbounded, so it is clear that the orbit of v is bounded $\Leftrightarrow a_2=a_3=a_4=0$ (a_1 arbitrary)$\Leftrightarrow v\in\operatorname{Ker} A$, which proves the first part of the theorem.

Now consider $x=(0,1,0,0)\in V\sim V_c$ and let U be any neighborhood of x in V (similar reasoning applies using any non-zero scalar $\lambda\neq 0$ in place of $\lambda=1$). For any infinite sequence $\{t_i: i=0,1,2,\dots\}$ in $\mathbf{R}$, let $U_i=\eta(t_i)U$. We will show that there exist $j\neq k$ such that $U_j\cap U_k\neq\varnothing$; consequently, no infinite sequence of transforms of U can be pairwise disjoint. Clearly we may assume that $t_0=0$, so $U_0=U$; transforming all sets by $\eta(-t_0)$ cannot alter disjointness relations. Without loss of generality we may also assume U has the form $U=\{(a_1, a_2, a_3, a_4)$: $|a_i|<\varepsilon$ for $i\neq 2$, and $|a_2-1|<\varepsilon\}$ for some ε with $0<\varepsilon<1/2$.

Let $\delta=12/\varepsilon$. If $|t_j|>\delta$ for some j, then $(0, 1, -6/t_j, 12/t_j^2)$ and $(0, 1, 6/t_j, 12/t_j^2)$ are both in U. Since

$$\eta(t_j)(0, 1, -6/t_j, 12/t_j^2)=(0, 1, 6/t_j, 12/t_j^2)$$

we get $(0, 1, 6/t_j, 12/t_j^2)\in U_j\cap U_0\neq\varnothing$. If $|t_j|\leqslant\delta$ for all j, then $\{t_j\}$ is bounded so that $|t_j-t_k|<\varepsilon$ for some pair $j\neq k$. Then $\eta(t_j-t_k)(0,1,0,0)=(t_j-t_k, 1, 0, 0)\in U$, so that

$$\eta(t_j)(0,1,0,0)=\eta(t_k)(t_j-t_k, 1, 0, 0)\in U_j\cap U_k\neq\varnothing. \qquad \text{Q.E.D.}$$

4.2 Corollary. *If $G=\mathbf{R}\times_\eta V$ as above, then G is simply connected nilpotent and*

$B(G)=Z(G)=V_c$. But there are points $x \in G \sim B(G)$ such that no neighborhood of x has infinitely many disjoint conjugates.

The proof is routine.

5. Proof of Theorem 1.4 (Step 1)

For reasons explained in section 2, we can restrict attention to connected Lie groups in which $K(G)$ is trivial (G without compact normal subgroups). Let $Z_G(N)$ be the centralizer of N in G; it is a closed characteristic subgroup in G, and is not necessarily connected. The purpose of this section is to prove the following lemma.

5.1 Lemma. *Let G be a connected Lie group with $K(G)_0$ trivial. Then there exists a layering of G that terminates with $Z_G(N)$.*

Proof. The nilradical N is closed and characteristic in G, and is simply connected by Lemma 3.1. Thus the Lie subgroups in the upper central series $N=N_m\supset \ldots \supset N_1 \supset N_0=\{e\}$,

$$N_m=N;\ N_{k-1}=\text{Lie subgroup generated by } [N, N_k],$$

are *closed* (all analytic subgroups are closed in a solvable simply connected group [4, p. 137]). They are characteristic in both N and G. Now define $H_k=\{x\in G\colon [N, x]\subset N_k\}$ for $0\leqslant k\leqslant m$; thus, $G=H_m\supset\ldots\supset H_1\supset H_0=Z_G(N)$. These sets are all closed in G since each N_k is closed. They are subgroups since: $[x, n]=xnx^{-1}n^{-1}\in N_k \Leftrightarrow \alpha_x(n)\equiv n(\text{mod } N_k)$, for all $n\in N$. Note that $H_k\supset N_k$ for all k. The inclusion $H_k\supset H_{k-1}$ need not be proper, even though $N_k\neq N_{k-1}$ for each k.

The subgroups H_k provide the desired layering of G. If $m=0$ then $G=Z_G(N)$ and there is nothing to prove. Otherwise, consider any k with $1\leqslant k\leqslant m$ and any point $u_0\in H_k\sim H_{k-1}$; again, if $H_k=H_{k-1}$ there is nothing to prove, so assume $H_k\neq H_{k-1}$. Then $[N, u_0]\subset N_k$ and $[N, u_0]\not\subset N_{k-1}$ since $u_0\notin H_{k-1}$, so there is an $n_0\in N$ such that $[n_0, u_0]\in N_k\sim N_{k-1}$. Since the N_i are closed, as are the H_i, we see that

There exists a relative neighborhood of u_0 in H_k such that $[n_0, u]\in N_k\sim N_{k-1}$ for all u this neighborhood. (2)

We will use powers of the inner automorphism $\alpha\colon g\to n_0 g(n_0)^{-1}$ to obtain the disjoint conjugates of a suitably chosen relative neighborhood of u_0. Let β be the inner automorphism induced by α on $\tilde{G}=G/N_{k-1}$. Let $\tilde{V}=N_k/N_{k-1}$ and $\tilde{M}=H_k/N_{k-1}$; then $\tilde{V}$ is a vector group since N is simply connected (Lemma 3.1). Write $\tilde{x}=\pi(x)$ for any $x\in G$, where $\pi\colon G\to\tilde{G}$ is the quotient homomorphism. For $u\in H_k$ near u_0, as in (2), we get

$$\alpha(u) = n_0 u (n_0)^{-1} u^{-1} u = [n_0, u]\, u \neq u \pmod{N_{k-1}}.$$

Since $\pi(\alpha(u)) = \beta(\pi(u))$ for all $u \in G$, and the image of a relatively open neighborhood of u_0 in H_k is a relatively open neighborhood of $\pi(u_0)$ in $\tilde{M}$, we see that:

$$\text{For all } \tilde{u} \text{ near } \tilde{u}_0 \text{ in } M,\ \beta(\tilde{u}) = [\tilde{n}_0, \tilde{u}]\cdot\tilde{u} \neq \tilde{u}. \tag{3}$$

Fix a relative neighborhood $\tilde{U}$ of $\pi(u_0)$ in $\tilde{M}$ such that (3) holds. Then $[\tilde{n}_0, [\tilde{n}_0, \tilde{u}]] \in \pi[N, N_k] \subset \pi(N_{k-1}) = \{\tilde{e}\}$, so that $\pi(n_0) = \tilde{n}_0$ commutes with all points in the set $[\tilde{n}_0, \tilde{U}]$ and products thereof. Thus, β leaves all such points fixed and

$$\begin{aligned} \beta(\tilde{u}) &= [\tilde{n}_0, \tilde{u}]\tilde{u} \\ \beta^2(\tilde{u}) &= \beta([\tilde{n}_0, \tilde{u}]\tilde{u}) = [\tilde{n}_0, \tilde{u}]\cdot\beta(\tilde{u}) = [\tilde{n}_0, \tilde{u}]^2\tilde{u} \\ &\ \ \vdots \\ \beta^p(\tilde{u}) &= [\tilde{n}_0, \tilde{u}]^p\tilde{u}, \end{aligned}$$

for $p = 1, 2, \ldots, \tilde{u} \in \tilde{U}$. Note that the "displacements" $[\tilde{n}_0, \tilde{u}]^p$ all lie in the vector group $\tilde{V} = N_k/N_{k-1}$. Define $\phi: \tilde{G} \to \tilde{G}$ via $\tilde{u} \to [\tilde{n}_0, \tilde{u}]$. Then ϕ is a continuous map of $\tilde{M}$ into $\tilde{V}$, since $\phi(\tilde{M}) = \pi[n_0, H_k] \subset \pi(N_k) = \tilde{V}$. Using additive notation in $\tilde{V}$, and noting that $\phi(\tilde{u}_0) \neq 0$ in $\tilde{V}$, we see that there exists a relatively open neighborhood A of $\phi(\tilde{u}_0)$ in $\tilde{V}$, and integers $n(1) < n(2) < \ldots$ such that the sets $n(k)A$ (scalar multiples of A) are pairwise disjoint for $k = 1, 2, \ldots$. Now replace $\tilde{U}$ above by any smaller *compact* relative neighborhood of $\tilde{u}_0$ in $\tilde{M}$ such that $\phi(\tilde{U}) \subset A$. Let $\tilde{W} = \tilde{V} \cap \tilde{U}\tilde{U}^{-1}$ (a symmetric neighborhood of zero in $\tilde{V}$). If we set $(1/2)\tilde{W} = \{(1/2)w : w \in \tilde{W}\}$, it is clear that the sets in $\tilde{V}$:

$$n(k)\,\phi(\tilde{U}) + \tfrac{1}{2}\tilde{W} = n(k)\left[\phi(\tilde{U}) + \frac{1}{2\,n(k)}\,\tilde{W}\right] \tag{4}$$

lie within $n(k)A$ for all large k (say $k \geq N_0$), because $n(k) \to +\infty$ and $\phi(\tilde{U})$ is compact in the open set A, and $\tilde{W}$ is also compact. Thus the sets $n(k)\ \phi(\tilde{U}) \times (1/2)\tilde{W}$ are disjoint for $k \geq N_0$.

Now examine the action of β in the β-invariant closed subset $\tilde{M} \subset \tilde{G}$; $\beta = \alpha_{\pi(n_0)} \in \mathfrak{J}(\tilde{G})$ and the conjugates $\{\beta^n(\tilde{U}) : n \in \mathbf{Z}\}$ lie in $\tilde{M}$ since $\tilde{U}$ is a relative neighborhood of $\tilde{u}_0$ in $\tilde{M}$. The conjugates $\beta^{n(k)}(\tilde{U})$ are disjoint for $k \geq N_0$; indeed, if $p > q \geq N_0$ give intersecting conjugates, then there would be points $u_1, u_2 \in U$ with

$$\beta^{n(p)}(u_1) = \phi(u_1)^{n(p)}u_1 = \phi(u_2)^{n(q)}u_2 = \beta^{n(q)}(u_2),$$

which would imply that

$$\phi(u_2)^{-n(q)}\phi(u_1)^{n(p)} = u_2(u_1)^{-1} \in \tilde{V} \cap \tilde{U}\tilde{U}^{-1} = \tilde{W} \subset \tfrac{1}{2}\tilde{W} - \tfrac{1}{2}\tilde{W}$$

(all within $\tilde{V}$), so that there would exist points $w_1, w_2 \in \tilde{W}$ with

$$n(p)\phi(u_1)+\tfrac{1}{2}w_1=n(q)\phi(u_2)+\tfrac{1}{2}w_2,$$

contrary to the disjointness of the sets (4). Thus, there are infinitely many disjoint conjugates $\beta^n(\tilde{U})$ in $\tilde{M}$. If $U=\pi^{-1}(\tilde{U})$, then the conjugates $\alpha^n(U)$ are disjoint, as in the proof of Lemma 2.1. Q.E.D.

5.2 COROLLARY. *If N is a connected nilpotent Lie group then $B(N)=\pi^{-1}(Z(N/K))$ where $K=K(N)$ is the maximal compact normal subgroup and $\pi\colon N\to N/K$ the canonical homomorphism. For finite central measures we have* supp $(\mu)\subset B(N)$.

It is worth noting that the connectedness of G is never really used in 5.1; the proof uses only the connectedness of the nilradical. For non-connected Lie groups, define $N(G)=$ the nilradical of G_0. This remark may be useful in later studies of central measures. It already yields the following result concerning non-connected groups.

5.3 COROLLARY. *Let G be a (not necessarily connected) Lie group whose connected component has no proper compact connected normal subgroups, so that $K(G_0)_0$ is trivial. Then* supp $(\mu)\subset Z_G(N)$ *for every finite central measure μ, where N is the nilradical of G.*

6. Proof of Theorem 1.4 (Step 2)

In this section we shall deal with the semi-simple part of G by examining the map $\pi\colon G\to G'=G/R$ where $R=\operatorname{rad}(G)$. Then G' is a semisimple Lie group and there exists a semisimple Lie subgroup $S\subseteq G$ such that $G=SR$ and $\pi|S$ is a local isomorphism. Notice that $K(G')$ may be non-trivial even if $K(G)$ is trivial.

For connected semisimple Lie groups, the unbounded conjugacy classes have been described in [14] and [10], respectively, where it is shown that

(i) $B(G')=Z(G')\cdot K(G')$

(ii) The orbit is uniformly unbounded for *every* point outside of $B(G')$.

Let $C=\pi^{-1}(B(G'))$; as in Lemma 2.1, it is obvious that the orbit of any point $x\in G\sim C$ is uniformly unbounded, so there is a one-step layering of G terminating with C, which is a closed characteristic subgroup of G.

If $H=Z_G(N)$, then by Lemma 2.2 there is a layering of G that terminates with the closed characteristic subgroup $A=H\cap C$. Our main observation is the following.

6.1 LEMMA. *Let G be a connected Lie group without proper compact normal subgroups. Then the connected component A_0 of the subgroup $A=H\cap C$ is the vector group $V=Z(N)$.*

This lemma will allow us to restrict our attention to the action of $\mathcal{J}(G)$ on A, rather than all of G. Furthermore, since each coset of A_0 is a copy of V, we will be able to reduce the proof of Theorem 1.4 to the study of certain affine actions on the vector space V. These problems will be treated in the next section.

Proof of 6.1. The closed subgroups C, $H = Z_G(N)$, A and their connected components C_0, H_0, A_0 are all characteristic in G. We first note that

$$\text{rad}\,(A_0) = V = Z(N). \tag{5}$$

Obviously $V \subset \text{rad}\,(A_0)$ since V is abelian, normal in G, and $C \supset R \supset V$, $H = Z_G(N) \supset V$. But rad (A_0) is normal in G; it is also connected and solvable. Thus, rad $(A_0) \subset R = \text{rad}\,(G)$. If rad (A_0) extends outside of N, then N and rad (A_0) generate a connected nilpotent Lie subgroup (since rad $(A_0) \subset A \subset H$ centralizes N) that is normal in R. This violates the definition of N. Thus, rad $(A_0) \subset N$, which implies that rad $(A_0) \subset A_0 \cap N \subset H \cap N = Z(N) = V$.

Now write $A_0 = S_1 \cdot V$ where S_1 is a semisimple Lie subgroup of A_0. Obviously S_1 commutes with V since $A_0 \subset H$ centralizes $N \supset V$. This forces S_1 to be *uniquely determined*, hence characteristic in A_0, because all other semisimple local cross sections are obtained from S_1 through conjugation by elements in rad (A_0). Thus S_1 has no nontrivial compact normal connected subgroups; these would lie within $K(S_1)$ and the latter would be characteristic in A_0, normal in G, and nontrivial, in violation of our hypotheses on G.

Now $A_0 = (H \cap C)_0 \subset H_0 \cap C_0$, so that $\pi(A_0) \subset (\pi(C))_0$. Furthermore $\pi | S_1$ is a homomorphism of S_1 onto a purely noncompact semisimple normal subgroup of $B(G')_0 \subset K(G')$. Thus $\pi(S_1) = \{e\}$, so that $S_1 = \{e\}$ and $A_0 = S_1 \cdot V = V$. Q.E.D.

7. Proof of Theorem 1.4 (Step 3): Action of $\mathcal{J}(G)$ on $A = Z_G(N) \cap C$

Let G be a Lie group and V a finite dimensional vector space. If $L: G \to GL(V)$ is a differentiable representation, the map $\beta: G \times V \to V$ given by $(g, v) = L(g)v$ will be called a *linear action*. If $T: G \to V$ is a differentiable map, then $\gamma: G \times V \to V$ given by $\gamma(g, v) = L(g)v + T(g)$ is an *affine action* (preserves convex sums in V).

Now assume G is a connected Lie group without proper compact normal subgroups. Since G is connected, each coset of A_0 in A is $\mathcal{J}(G)$-invariant. We denote these cosets by $a_i V$, $a_i \in Z_G(N)$. By the previous results, $B(G) = \{a \in A$: $\mathcal{J}(G)$-orbit has compact closure in $A\}$. Since V is a vector space, the action of G on V given by $v \to gvg^{-1}$ is a representation. An easy calculation shows that for each a_i the affine action τ_i: $G \times V \to V$ given by $\tau_i(g, v) = gvg^{-1} + [a_i^{-1}, g]$ is G-equivariant with the action β_i: $G \times A_i \to A_i$ given by $\beta_i(g, a_i v) = g a_i v g^{-1}$ via the map ψ: $V \to A_i$ where $\psi(v) = a_i v$. Thus we are reduced to studying affine actions on V.

For any connected subgroup $G' \subset G$ we write $A_{i,c}\,(G')$ for the elements of A_i with bounded G'-orbits (or just $A_{i,c}$ if $G' = G$). To prove Theorem 1.4 it suffices to show

7.1 Lemma. *Let G be a connected Lie group with $K(G)$ trivial. Then in each coset $C = A_t$ there exists a layering $C = C_m \supset \ldots \supset C_0 = A_{t,c}$.*

To prove the lemma, we invoke the following elementary result whose proof we omit.

7.2 Lemma. *If $\beta: G \times V \to V$ is an affine action there is a linear action $\gamma: G \times W \to W$ where $W = V \oplus \mathbf{R}$, such that* (i) *the hyperplane $V^* = \{(v, 1): v \in V\}$ is G-invariant* (ii) *the actions $\gamma: G \times V^* \to V^*$ and $\beta: G \times V \to V$ are equivariant under the identification*

$$\psi: V \to V^*, \quad \text{where} \quad \psi(v) = (v, 1).$$

If V_c, W_c are the elements with precompact orbits, then $\psi(V_c) = W_c \cap V^*$.

Hence Theorem 1.4 is reduced to proving the following result for *linear* actions.

7.3 Theorem. *Let G be a connected Lie group and $G \times V \to V$ a linear action of G on a finite dimensional vector space V. If V_c denotes the set of elements with bounded orbits, then there exist G-invariant subspaces $V = V_m \supset \ldots \supset V_c$ such that each $x \in V_k \sim V_{k-1}$ has a relative neighborhood in V_k with infinitely many disjoint G-transforms.*

Theorem 7.3 will be proved in section 8.

8. Linear actions of G on a vector space

We now take up the proof of Theorem 7.3 (and so, of Theorem 1.4). We begin with the special cases in which $G = \mathbf{R}$ or G is a connected semisimple Lie group.

8.1 Proposition. *Let $\eta: \mathbf{R} \times V \to V$ be a linear action on a vector space (a 1-parameter transformation group). Let V_c be the subspace of points with bounded orbits. There exists a layering of V that terminates with V_c.*

We prove Proposition 8.1 in a series of lemmas. Let A be the infinitesimal generator of the 1-parameter group, i.e., $\eta(t) = e^{tA}$. There is a normal form of A which facilitates our analysis; unfortunately, it does not seem to appear explicitly in the literature, so we include a proof.

8.2 Lemma. *Given any linear operator A on a real vector space V we can express A as a sum $A = A_r + A_t + N$ of operators on V, and decompose V as a direct sum of subspaces $V = V_1 \oplus \ldots \oplus V_m$, so that* (i) *The operators A_r, A_t, N commute pairwise and leave each V_k invariant.* (ii) *A_r is diagonalizable and acts on V_k as scalar multiplication by a real scalar x_k (the real part of an eigenvalue λ_k).* (iii) *A_t is skew-symmetric (with respect to a suitable basis in each V_k).* (iv) *N is nilpotent.*

Proof. Let $A = A_s + N$ be the Jordan decomposition of A. For each complex number z let $X_z \subset V \oplus \mathbf{C}$ be the corresponding eigenspace of A_s. It is easily seen that $X_z + X_{\bar{z}}$ is the complexification of its real part $X_{z,\bar{z}} = (V + i0) \cap (X_z + X_{\bar{z}})$. Thus V is the direct sum of the various $X_{z,\bar{z}}$. Let A_r be the semisimple operator which acts as the scalar Re z on $V_{z,\bar{z}}$. Now $A_i = A_s - A_r$ has the desired skew symmetry on $X_{z,\bar{z}}$ since it acts on $X_z + X_{\bar{z}}$ as a pure imaginary scalar when complexified. Finally N commutes with A_r because it leaves invariant X_z, $X_z + X_{\bar{z}}$, and the real part $X_{z,\bar{z}}$, on which A_r is a scalar. Thus N commutes with A_i. Q.E.D.

In each V_k take a basis so that A_i is skew-symmetric and impose the corresponding inner product norm. On V introduce a norm compatible with the direct sum $V = V_1 \oplus \ldots \oplus V_m$; thus if $v = v_1 + \ldots + v_m$, we take $\|v\|^2 = \|v_1\|^2 + \ldots + \|v_m\|^2$. For each $t \in \mathbf{R}$, e^{tA_i} has orthogonal matrices for its diagonal blocks, so

$$\|e^{tA_i}(v)\| = \|v\| \quad \text{for all} \quad v \in V. \tag{6}$$

Since N is nilpotent and e^{-tN} is the inverse of e^{tN}, we also have

$$\|e^{tN}(v)\| \geq \frac{\|v\|}{p(t)} \text{ all } v \in V, \quad \text{all } t \in \mathbf{R}, \tag{7}$$

where $p(t) = 1 + |t|\,\|N\| + \ldots + |t|^s\,\|N\|^s/s!$ (s a power such that $N^s = 0$); indeed, $\|v\| = \|e^{-tN}e^{tN}v\| \leq \|e^{-tN}\|\,\|e^{tN}v\| \leq p(t)\|e^{tN}v\|$.

8.3 LEMMA. *If $v \notin \operatorname{Ker}(A_r)$, there is a neighborhood in V with infinitely many disjoint transforms under $\eta(\mathbf{R})$. In particular, $V_c \subset \operatorname{Ker}(A_r)$.*

Proof. Write $v = v_1 + \ldots + v_m$, ($v_k \in V_k$). Since $v \notin \operatorname{Ker}(A_r)$, there is an index k such that Re $(\lambda_k) = x_k \neq 0$ and $\|v_k\| = \delta > 0$. Take any $K > 0$ such that $0 < \delta < K$ and examine the neighborhood of the form $W = \{w = w_1 + \ldots + w_m : \|w_j\| \leq K \text{ all } j, \text{ and } \|w_k\| \geq \delta/2\}$. To see that W has the desired properties, assume $\{t_1 \ldots t_n\}$ have been chosen so that $\eta(t_j)W$ are pairwise disjoint for $1 \leq j \leq n$, and find a t_{n+1} such that $\eta(t_{n+1})W$ is disjoint from these. [Note that t_1 may be chosen at random.] Let M be any bound for the norms of w in $\bigcup_{j=1}^{n} \eta(t_j)W$. Now if $w = w_1 + \ldots + w_m \in W$ we have

$$e^{tA_r}(w) = \sum_{j=1}^{m} e^{tx_j} w_j \quad (\text{writing } \lambda_k = x_k + iy_k)$$

and since the subspaces V_j are mutually orthogonal and N-invariant,

$$\|e^{tA}(w)\|^2 = \|e^{tA_t}e^{tA_r}e^{tN}(w)\|^2 = \left\|e^{tN}e^{tA_r}\left(\sum_{j=1}^{m} w_j\right)\right\|^2$$

$$= \left\|e^{tN}\left(\sum_{j=1}^{m} e^{tx_j}w_j\right)\right\|^2 = \sum_{j=1}^{m} e^{2tx_j}\|e^{tN}w_j\|^2$$

$$\geqslant e^{2tx_k}\|e^{tN}(w_k)\|^2 \geqslant \frac{e^{2tx_k}}{p(t)^2}\|w_k\|^2 \geqslant e^{2tx_k}\frac{\delta^2}{4p(t)^2}$$

for all $w \in W$, $t \in \mathbf{R}$. Since $x_k \neq 0$ we can obviously choose $t = t_{n+1}$ so that $\|e^{tA}(w)\| > M$ for all $w \in W$; thus the sets $e^{t_jA}W$ are disjoint.

Proof of Proposition 8.1. If V' is an A-invariant subspace of V, then $V'_c = V' \cap V_c$. Therefore, by induction, it suffices to show that if $V \neq V_c$ there is an A-invariant subspace V' with $V \neq V' \supset V_c$ such that each $v \in V \sim V'$ has a neighborhood with infinitely many disjoint transforms $\eta(t)U = e^{tA}U$. If $\operatorname{Ker}(A_r) \neq V$, then $V' = \operatorname{Ker}(A_r)$ is invariant and points $v \in V \sim V'$ have uniformly unbounded orbits (hence $V' \supset V_c$) as required. Therefore, we may assume that $A_r = 0$ and $A = A_t + N$. If $N = 0$ also, then $\eta(t) = e^{tA_t}$ is orthogonal for all t and every orbit in V is bounded, so that $V = V_C$ and there is nothing to prove; thus, assume $A = A_t + N$ where $N \neq 0$.

Consider the kernels $\{0\} \subsetneqq \operatorname{Ker} N \subsetneqq \dots \subsetneqq \operatorname{Ker} N^{m-1} \subsetneqq \operatorname{Ker} N^m = V$, ($m \geqslant 2$ since $N \not\equiv 0$). The proper subspace $V' = \operatorname{Ker} N^{m-1}$ will satisfy our requirements. Let $V'' = \operatorname{Ker} N^{m-2}$ and note that (i) $\{0\} \subseteq V'' \subsetneqq V' \subsetneqq V$, (ii) each space $\operatorname{Ker} N^k$ is invariant under A, A_t, N. In demonstrating that any point $v_0 \in V \sim V'$ has uniformly unbounded orbit, we may assume $V'' = \{0\}$. Otherwise we could pass to the induced operator $\tilde{N}$ on $\tilde{V} = V/V''$ (for which $m = 2$ and $\tilde{V}'' = \operatorname{Ker} \tilde{N}^{m-2} = \{0\}$), then produce disjoint transformed neighborhoods of $\tilde{v}_0$, and finally lift things back to V.

Assume $V'' = \{0\}$. If V is equipped with an inner product such that A_t is skew-symmetric, e^{tA_t} is orthogonal so that $\|e^{tA}v\| = \|e^{tN}v\|$ for $v \in V$. Clearly $v_0 \in V \sim V'$ implies that $Nv_0 \in V' \sim V''$, so that $Nv_0 \neq 0$; however, $N^2 = 0$ on V, so that

$$e^{tN}(v) = v + tN(v) \quad \text{for all} \quad t \in \mathbf{R}, \quad \text{all} \quad v \in V. \tag{8}$$

Pick any bounded open neighborhood W of $N(v_0)$ that is bounded away from zero. We may choose $t(1) < t(2) < \dots$ increasing toward infinity so that the norms of the points in the sets $t(k)W$ lie in disjoint intervals: if $\|w\| \in [r_0, s_0]$ for $w \in W$ $(0 < r_0 < s_0)$, then for $w \in t(k)W$ we get $\|w\| \in t(k)[r_0, s_0] = [r_k, s_k]$. Now choose any compact neighborhood U of v_0 such that $N(U) \subset W$. By (8) we get

$$e^{t(k)N}(U) \subset U + t(k)N(U) = t(k)\left[\frac{1}{t(k)}U + N(U)\right];$$

since $t(k)\to+\infty$, W is open, and U compact, we get

$$\frac{1}{t(k)}U+N(U)\subset W \quad\text{and}\quad e^{t(k)N}(U)\subset t(k)W$$

for all large k, say $k\geqslant k_0$. This insures that the sets $e^{t(k)A}(U)$ are disjoint for $k\geqslant k_0$, because if $u\in U$ we have

$$\|e^{tA}(u)\|=\|e^{tN}(u)\|\in[r_k, s_k] \quad\text{(disjoint intervals).}$$

This proves the proposition. Q.E.D.

The actions of semisimple groups are less complicated.

8.4 Proposition. *Let G be a connected semi-simple Lie group and $\varrho: G\to GL(V)$ a real linear representation. Let V_c be the subspace of points with bounded orbits. Then every point $v\in V\sim V_c$ has a neighborhood in V with infinitely many disjoint G-transforms. If G has no compact factors (i. e., if $K(G)$ is trivial) then $V_c=\{v\in V: G(v)=v\}$.*

Proof. Let $K=K(G)$; there is a semi-simple normal Lie subgroup S such that (i) S has no compact factors, and (ii) $G=SK$, a local direct product with commuting factors: $[S, K]=\{e\}$. Clearly $V_c(G)=V_c(S)$ since all points of V have compact orbits under K and the action $G\times V\to V$ is jointly continuous. Thus it suffices to prove that 8.4 is valid when G is a connected semi-simple Lie group without compact factors.

Since we are assuming G has no compact factors G is generated by Lie subgroups locally isomorphic to $SL(2, \mathbf{R})$, cf. Serre [13; Ch. VI, Thm. 2]. Therefore, it suffices to take $G=SL(2, \mathbf{R})$ and to show that if $v\in V$ is such that $\varrho(g)v\neq v$ for some $g\in G$, then the G-orbit of v is uniformly unbounded. Let $\mathfrak{g}=\mathfrak{sl}(2, \mathbf{R})$ and let $d\varrho: \mathfrak{g}\to\operatorname{End}(V)$ be the corresponding representation of $\mathfrak{g}$ on V.

Since G is generated by one-parameter subgroups $\{\exp(tX): X\in\mathfrak{g}\}$, and since $e^{td\varrho(X)}=\varrho(\exp(tX))$ for $X\in\mathfrak{g}$, it follows that: if $\varrho(g)v\neq v$ for some $g\in G$, this implies that $d\varrho(X)v\neq 0$ for some $X\in\mathfrak{g}$. However, $\mathfrak{g}$ is generated as a Lie algebra by the matrices

$$X_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad\text{and}\quad X_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

so it follows that: $d\varrho(X_1)v\neq 0$ or $d\varrho(X_2)v\neq 0$. Now, $d\varrho(X_i)$, $i=1, 2$, is diagonalizable (over $\mathbf{R}$) with all its eigenvalues real [13; Ch. IV, Th. 1]. Each one-parameter group $\eta_i(t)=\exp(td\varrho(X_i))$ lies within $\varrho(G)$ and has infinitesimal generator $A_j=d\varrho(X_j)$, $j=1, 2$. In terms of the decomposition $A=S+N=(A_r+A_i)+N$, discussed in Lemma 8.2, we have $A=A_r$ and $A_i=N=0$ in each case, and our conditions on v mean precisely that $v\notin\operatorname{Ker}A_1$ (resp.

$v \notin \operatorname{Ker} A_2$). If $v \notin \operatorname{Ker} A_j$, $j=1, 2$, the result follows by applying 8.3 to the one-parameter group $\eta_j(\mathbf{R})$. Q.E.D.

In order to combine these results we must be able to decide whether the G-orbit of a vector v is bounded in terms of boundedness under the separate actions of generating subgroups $G_i \subset G$.

8.5 Lemma. *Let G_1, G_2 be analytic subgroups of a Lie group G, with G_2 normal in G. Suppose that $G=G_1G_2$. If $G \times V \to V$ is a linear action on a vector space V_c let $V_c = V_c(G)$ (resp. $V_c(G_i)$) be the subspaces of elements with bounded orbits under the action of G (resp. G_i). Then $V_c = V_c(G_1) \cap V_c(G_2)$.*

Proof. The inclusion ($\subset$) is obvious. Conversely, write $V_0 = V_c(G_1) \cap V_c(G_2)$ and $W = V_c(G_2)$. The subspace W is G_1 and G_2 invariant; G_2 invariance is clear, and G_1 invariance follows from normality of G_2: $G_2(g_1 v) = g_1 \cdot g_1^{-1} G_2 g_1(v) = g_1 G_2(v)$ is precompact if $G_2(v)$ is. We start by proving that $G_2|W$ is precompact in $\operatorname{End}_{\mathbf{R}}(W)$. We take any norm $\|v\|$ on W and refer to the operator norm $\|T\| = \sup\{\|T(v)\|: \|v\| \leqslant 1, v \in W\}$ for endomorphisms. By hypothesis we have $\sup\{\|g(v)\|: g \in G_2\} < +\infty$, for each $v \in W$. By the uniform boundedness principle, we must have: $\sup\{\|g\|: g \in G_2\} < +\infty$, so G_2 gives a norm bounded set $G_2|W$ of linear operators on W, which must be precompact due to finite dimensionality. The function $f(T) = \det T$ must be bounded on $G_2|W$; since $G_2|W$ is a group, $f(T)$ must also be bounded away from zero. Therefore the closure H of $G_2|W$ is a compact subgroup in $GL(W)$.

Now consider the action of G on a point $v \in V_0 (\subset W)$; $G_1(v)$ is precompact and $G_1(g_2(v)) = \{g_1 g_2(v) = \alpha_{g_1}(g_2) \cdot g_1(v): g_1 \in G_1\} \subset H(G_1(v))$. The latter set is precompact in W because H is compact and $GL(W) \times W \to W$ is a jointly continuous map. Thus $v \in V_0$ implies that $G(v) = (G_1 \cdot G_2)v = \cup\{G_1(g_2 v): g_2 \in G_2\} \subset H(G_1 v)$, so that $v \in V_c$. Hence $V_0 = V_c(G_1) \cap V_c(G_2) \subset V_c$. Q.E.D.

We note that the above result also holds for affine actions. Now we combine previous lemmas; our basic tool for this is the following observation.

8.6 Lemma. *Let $G \times V \to V$ be a linear action of a connected Lie group on a vector space. Let A be a G-invariant subspace of V and let H be a subgroup of G. If there is an H-invariant subspace V' such that $V \neq V' \supset A$ and every point $v \in V \sim V'$ has a neighborhood with infinitely many disjoint H-transforms, then there is a G-invariant subspace W with $V \neq W \supset A$ such that every $v \in V \sim W$ has a neighborhood with infinitely many disjoint G-transforms. We may take $W \subset V'$.*

Proof. There is nothing to prove if $V = A$. If $V \neq A$ take the H-invariant subspace V' and form the G-invariant subspace $W = \bigcap\{gV': g \in G\}$. Since A is G-invariant, $V \neq V' \supset W \supset A$.

Now if $v \in V \sim W$, there is a $g \in G$ such that $v \notin gV'$, which means that $v_0 = g^{-1}v \notin V'$. By hypothesis, there is a neighborhood $U \subset V \sim V' \subset V \sim W$ of v_0 with infinitely many translates $h_i(U)$, $h_i \in H$; thus, $\tilde{U} = g(U)$ is a neighborhood of v in $V \sim W$, and the G-translates $h_i g^{-1}(\tilde{U}) = h_i(U)$ are disjoint.

8.7 LEMMA. *Let $G = G_1 \cdot G_2$ where G_1 is normal in G. If there exist layerings* (i) *under the action of G_1 terminating with $V_c(G_1)$, and* (ii) *under the action of G_2 terminating with $V_c(G_2)$, then there exists a layering under the action of G (via G-invariant subspaces) terminating with $V_c = V_c(G)$.*

Proof. This is trivial if $\dim(V) = 0$. Assuming $\dim(V) \geq 1$, there is nothing to prove if $V_c = V$. Otherwise, let $V \supset V_1 \supset \ldots \supset V_r = V_c(G_1)$ by the layering under the action of G_1. If $V_c(G_1) \neq V$, then $V_1 \neq V$ and by 8.6 there is a G-invariant subspace V' such that $V_1 \supset V' \supset V_c(G)$ such that every point $v \in V \sim V'$ has a neighborhood with infinitely many disjoint G-transforms. Our hypotheses remain true for the restricted actions of G_1 and G_2 on V', so by induction we get a layering of V under the action of G which terminates with V_c.

If $V_c(G_1) = V$, then $V_c(G_2) = V_c(G_1) \cap V_c(G_2) = V_c(G) \neq V$, so the layering under the action of G_2, $V \supset W_1 \supset \ldots \supset W_s = V_c(G_2) = V_c$, has $W_1 \neq V$. By 8.6, there is a G-invariant subspace $W' \subset W_1$ such that every $v \in V \sim W'$ has a neighborhood with infinitely many disjoint G-transforms. Now apply the induction hypothesis to the actions of G_1 and G_2 on W'. Q.E.D.

To prove Theorem 7.3 for a linear action of an arbitrary connected group G, first replace the action of G by the lifted action of its simply connected covering group. The covering group can be written $G = \prod_{k=1}^{n} G_k$ where the G_k are closed subgroups such that G_n is semisimple, $\dim G_k = 1$ for $k < n$, and G_k normalizes $\prod_{j<k} G_j$. By 8.5 we have $V_c = \bigcap_k V_c(G_k)$. Now apply 8.1, 8.4, and 8.7 repeatedly.

9. Refined description of $B(G)$

Now that Theorem 1.4 is available to handle the uniformity properties of unbounded conjugacy classes, we turn to the question of giving a more detailed description of the subgroup $B(G)$ in certain cases. In [14] Tits introduced the notion of an *automorphism of bounded displacement* as follows. If G is a locally compact group and $\alpha \in \mathcal{A}(G)$, the automorphism group of G, one says that α is of bounded displacement, or is a *bd* automorphism, if $x^{-1}\alpha(x)$ lies in a fixed compact subset of G for all $x \in G$. In addition, one observes that an inner automorphism α_g is a *bd* automorphism if and only if $g \in B(G)$. In [14] Tits has considered the question of characterizing $B(G)$, and more generally, finding the *bd* automorphisms when G is a Lie group. He proved, among other things, a result (Théorème 1) which implies that if G is a connected, simply connected nilpotent Lie group, it has no nontrivial *bd* automorphisms. Here we extend this conclusion to certain other classes of groups.

9.1 THEOREM. *Let G be a connected Lie group whose radical R has no nontrivial bd automorphisms. If G/R has no compact factors, then G has no non-trivial bd automorphisms.*

This result extends Tits' Corollaire (2), where R is assumed to be nilpotent and simply connected. We need two lemmas.

9.2 LEMMA. *Let G be a connected Lie group having no non-trivial central torus. Then $K(N)$ is trivial, where N is the nilradical of G. Every bd automorphism of G is an inner automorphism by some $g \in B(G)$, and $B(G) = Z(G) \cdot V$ where V is a vector subspace in $Z(N)$.*

Proof: We claim that $K(N)_0$ is central in G. For if $X \in \mathfrak{k}$, the Lie algebra of $K(N)_0$, then ad X is nilpotent so that $\text{Exp}\,(\mathbf{R}\,\text{ad}\,X) = \text{Ad}\,(\exp\,(\mathbf{R}X)) \cong \mathbf{R}$, and therefore cannot be compact unless ad $X = 0$. Hence ad $X = 0$ for all $X \in \mathfrak{k}$. The rest of the lemma now follows immediately from Tits' Theoreme (1) and Theoreme (3). Q.E.D.

9.3 LEMMA. *Let R be a solvable Lie group with no non-trivial bd automorphisms. Then R contains no non-trivial central torus.*

Proof. Suppose $T \subset Z(R)$ were a torus, which we can assume to have dimension one. Define a homomorphism ϕ: $R \to T$ without fixed points other than the unit in R, as follows. If $R = T$, take any non-trivial homomorphism which is not the identity. If $\dim R > 1$, then there is a connected normal subgroup $R_1 \supset T$ of codimension one. There is also an onto homomorphism ϕ_1: $R/R_1 \to T$. If π: $R \to R/R_1$ is the quotient map, then $\phi = \phi_1 \circ \pi$: $R \to T$ is a homomorphism without fixed points other than the unit. Then θ: $g \to g \cdot \phi(g)^{-1}$ is a non-trivial *bd* automorphism on R.

Proof of 9.1. By Lemma 9.3, R and therefore G, contains no central torus. By Lemma 9.2, every *bd* automorphism is an inner automorphism by some $g \in V$. Now $V \subset Z(R)$ since $B(R) = Z(R)$ by hypothesis. Finally, the action of G/R on V by conjugation must be trivial, by 8.4, since G/R has no compact factors. Hence $V \subset Z(G)$. Q.E.D.

9.4 THEOREM. *Let G be a simply connected Lie group which is either* (i) *solvable of type* (E), *or* (ii) *complex analytic. Then G has no non-trivial* (*real analytic*) *automorphisms of bounded displacement. If (iii) G is any connected complex analytic linear group, then G has no non-trivial complex analytic automorphisms of bounded displacement (hence no inner bd automorphisms).*

Proof. In (i) and (ii) G has no central torus T, since $T \subset N$ would contradict the assumption that G is simply connected (recall 3.1). Hence by Lemma 9.2 it suffices to show that $V \subset Z(G)$.

In case (i), type (E) is defined as in [2], [12]; then ad X for $X \in \mathfrak{g}$ cannot have non-zero pure imaginary eigenvalues. Identifying V with its Lie algebra $\mathfrak{v}$, the orbit of a point $v \in V$ under $\mathfrak{J}(G)$ corresponds to the orbit of $v \in \mathfrak{v}$ under Exp (R ad X). Since this orbit is bounded, it must be trivial for each $X \in \mathfrak{g}$ in view of the eigenvalue condition. Thus the action of G by conjugation on V is trivial, so that $V \subset Z(G)$.

In case (ii) the action of G on V must also be trivial since the image of a complex one-parameter group of linear maps cannot be bounded unless it is a point.

Case (iii) could be handled similarly, but we refer the reader to a direct proof: see C. Sit, Ph. D. Thesis, CUNY Graduate Center (to appear). Q.E.D.

Theorem 9.4 also applies if G is a simply connected solvable linear group which is real algebraic, since such groups are of type (E).

Remark. The conclusion of (i) fails if G is not of type (E); (ii) fails if G is not simply connected, and (iii) fails in the case of real analytic automorphisms.

(i) Let G be the simply connected covering group of the group of Euclidean motions in the plane. If x is a non-trivial element of $[G, G]$ then the conjugacy class of x is a circle in the plane $[G, G]$, and so is compact. But $[G, G] \cap Z(G) = \{e\}$, and $Z(G) \neq \{e\}$.

(ii) Consider the complex Heisenberg group $N_3(\mathbf{C})$ of 3×3 complex upper triangular matrices with 1's on the diagonal. Let $G = N_3(\mathbf{C})/Z(N_3(\mathbf{C}))$. Then $[G, G]^-$ is compact, and in particular $B(G) = G$.

(iii) Let $G = GL(1, \mathbf{C}) = \mathbf{C}^*$, the multiplicative group of non-zero complex numbers. As a real analytic group, $G \cong \mathbf{R} \times T$, and the map $(r, t) \to (r, 1/t)$ is a nontrivial real analytic *bd* automorphism.

10. Remarks

All of $B(G)$ is needed to support central measures, so that Theorem 1.5 is the best possible result.

10.1 THEOREM. *Let G be any connected locally compact group. Given any $x \in B(G)$, there is a finite central measure μ such that $x \in$ supp (μ).*

We omit the proof, which is fairly routine.

Theorem 1.5 may be used to study central idempotent measures on G, those central measures μ such that $\mu * \mu = \mu$. Since $B(G) \supset$ supp (μ), we may apply results from [8] to $B(G)$

to prove that an *idempotent* central measure μ is, in fact, supported on $K(G)$. This observation allows one to determine all central idempotent measures on a connected locally compact group, extending earlier work in [1], [6], [8], [9], [10], [11]. These results will be published elsewhere.

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Note added in proof: Applications of the methods developed in this paper will appear in the following notes by the authors. (1) Central idempotent measures on connected locally compact groups, *J. Functional Analysis*, 15 (1974) 22–32. (2) Compactness of certain homogenous spaces of finite volume, *Amer. J. Math.*, (to appear, 1974). (3) Automorphisms, orbits, and homogenous spaces of non-connected Lie groups, (in preparation).

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