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UNIQUE CONTINUATION AND A SCHWARZ REFLECTION PRINCIPLE FOR ANALYTIC SETS

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§ 0. Introduction.

This paper deals with unique continuation of holomorphic functions at boundary points. Our first result (Theorem 1) shows that if h is a holomorphic function in a neighborhood of 0 in the complex upper half plane, continuous up to the boundary, vanishing of infinite order at 0, and mapping the reals into a half space, then h must vanish identically.

Theorem 1 is then used to sharpen and simplify previous results of unique continuation in complex analysis. In a joint work with Alinhac [2], the authors showed that any vector valued holomorphic function in a neighborhood of 0 in the complex upper half plane, continuous up to the boundary, vanishing of infinite order at 0, and mapping the reals into M, a totally real manifold in $\mathbf{C}^n$ of class C^2, must vanish identically. (See also Bell-Lempert [3] for a different proof.) In recent work, Huang and Krantz [6] were able to weaken the assumption in the result above by replacing C^2 by $C^{1,\alpha}$ with $\alpha > 0$. Our Theorem 2 deals with a holomorphic vector valued function mapping the reals into a subset of $\mathbf{C}^n$ of the form $\{|\Im z| \le \beta |\Re z|\}$ with $0 \le \beta < 1$. As a consequence (see Corollary (2.4)), we obtain a very simple proof of the unique continuation result mentioned above when the totally real manifold M is only of class C^1. (In particular this gives a completely different proof of a result of Alexander [1] when M is a C^1 curve.) Using a classical theorem of

Dirichlet, we show (Theorem 3) that unique continuation also holds when the function h maps the reals into a subset of the plane concentrated near finitely many rays.

We also give applications of Theorem 3 to the case where the holomorphic function h defined in a neighborhood of 0 in the upper half plane maps the reals into a real analytic subset of the plane. In this case, we first show (Theorem 4) that if h vanishes of infinite order at 0, then it must vanish identically. Then, in Theorem 5, we show that if h is of class C^∞ up to the boundary, it extends holomorpically across the real line. Note that if the real analytic set is a (smooth) real analytic curve, and hence can be mapped by a local biholomorphism, into the real line, then the latter result follows from the classical Schwarz reflection principle.

For the proof of Theorem 1 we have borrowed some ideas from the proof of Theorem 3.4 in [6], which deals with unique continuation of smooth functions mapping the reals into a positive hypersurface in C^n.

§1. Unique continuation for holomorphic functions mapping the real line into a half plane.

Let $\Omega \subset \mathbf{C}$ be a connected open neighborhood of 0. Denote by Ω^+ (resp. $\overline{\Omega}^+$) the set of points $z \in \Omega$ with $\Im z > 0$ (resp. $\Im z \geq 0$). For $\phi \in L^2(\mathbf{R})$, we denote by $K(\phi)$ the Hilbert transform of ϕ given by

$$K\phi(x) = \lim_{\epsilon\to 0} \frac{1}{\pi} \int_{|y|>\epsilon} \frac{\phi(x-y)}{y} dy. \tag{1.1}$$

THEOREM 1. *Let h be a continuous function in $\overline{\Omega}^+$, holomorphic in Ω^+ satisfying the following properties:*

(i) h vanishes of infinite order at 0, i.e. for every N there exists C_N such that

$$|h(z)| \leq C_N |z|^N, \quad z \in \Omega^+.$$

(ii) $\Re h(x) \geq 0, \quad x \in \Omega \cap \mathbf{R}$.
Then $h \equiv 0$ in Ω^+.

PROOF. We write $h(z) = u(z) + iv(z)$, where u and v are real valued. Without loss of generality, we may assume that $\Omega \cap \mathbf{R}$ is an open interval $I = (-r, r)$. Let $\chi \in C_0^\infty(I)$ with $\chi(x) \equiv 1$ for $|x| \leq \frac{r}{2}$, and $0 \leq \chi(x) \leq 1$. For $x \in I$ we write

$$\chi(x)h(x) = \chi(x)u(x) + i\chi(x)v(x) = f(x) + ig(x), \tag{1.2}$$

where f and g are continuous with compact support and real-valued. Note that by the hypothesis f and g vanish of infinite order at 0, and f is nonnegative.

We need the following lemmas.

LEMMA (1.3). *With the notation (1.1) and (1.2), for every $n = 0, 1, 2, \dots$ the function*

$$a_n(x) = g(x)x^{-n} - K(f(y)y^{-n})(x) \tag{1.4}$$

is real analytic in an open neighborhood of 0 in $\mathbb{R}$, and hence the function $K(f(y)y^{-n})(x)$ is continuous in the same neighborhood.

PROOF. First note that for any n, $f(x)x^{-n}$ is continuous with compact support in $\mathbb{R}$, and hence is in $L^2(\mathbb{R})$. Therefore, $K(f(y)y^{-n})(x) \in L^2(\mathbb{R})$ (see e.g. [8]). On the other hand, the function $f(x)x^{-n} + iK(f(y)y^{-n})(x) \in L^2(\mathbb{R})$ has a holomorphic extension to the upper half plane $\mathbb{C}^+$ given by

$$k_n(z) = \frac{1}{i\pi}\int_{\mathbb{R}} \frac{f(x')}{(z - x')x'^n}\,dx'. \tag{1.5}$$

Note also that the function $p_n(x) = f(x)x^{-n} + iK(f(y)y^{-n})(x) - \chi(x)h(x)x^{-n}$ extends to a holomorphic function in the upper half space near $x = 0$ and is purely imaginary for $x \in \mathbb{R}$ near 0, since $\chi \equiv 1$ near 0 and $h(x)x^{-n}$ extends holomorphically to Ω^+. Hence, by the Schwarz reflection principle, $p_n(z)$ extends holomorphically to a full neighborhood of 0 in $\mathbb{C}$. Since $a_n(x) = -\Im p_n(x)$ it follows that $a_n(x)$ is real analytic (and therefore continuous) in a neighborhood of 0 in $\mathbb{R}$. Since $g(x)x^{-n}$ is continuous, so is $K(f(y)y^{-n})(x)$, which proves the lemma. ♠

LEMMA (1.6). *If $\alpha(x)$ and $\alpha(x)x^{-1}$ are both continuous with compact support on $\mathbb{R}$, then*

$$K(\alpha)(x) + \frac{1}{\pi}\int_{\mathbb{R}} \frac{\alpha(y)}{y}\,dy = xK\left(\frac{\alpha(y)}{y}\right)(x), \tag{1.7}$$

where the equality is in the sense of L^2 functions.

PROOF. Using definition (1.1) of K, we write

$$K(\alpha)(x) + \frac{1}{\pi}\int_{\mathbb{R}} \frac{\alpha(y)}{y}\,dy = \frac{1}{\pi}\lim_{\epsilon\to 0}\Big[\int_{|x-y|>\epsilon} \frac{\alpha(y)}{(x-y)}\,dy + \int_{|x-y|>\epsilon} \frac{\alpha(y)}{y}\,dy\Big]. \tag{1.8}$$

Since $(x-y)^{-1} + y^{-1} = xy^{-1}(x-y)^{-1}$, the right hand side of (1.8) becomes

$$\frac{1}{\pi}\lim_{\epsilon\to 0} x\int_{|x-y|>\epsilon}\frac{\alpha(y)}{y(x-y)}dy = \frac{1}{\pi}x\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}\frac{\alpha(y)}{y(x-y)}dy = xK(\frac{\alpha(y)}{y})(x). \tag{1.9}$$

This proves the lemma.♠

We may now complete the proof of Theorem 1. If $f(x)$ and $g(x)$ are defined by (1.2), we obtain from Lemma (1.6) applied to $\alpha(x) = f(x)/x^n$

$$K(\frac{f(y)}{y^n})(x) + \frac{1}{\pi}\int_{\mathbb{R}}\frac{f(y)}{y^{n+1}}dy = xK(\frac{f(y)}{y^{n+1}})(x), \tag{1.10}$$

for $n = 0, 1, 2, \ldots$. Let $a(x) = a_0(x)$ be given by (1.4). Since $a(x)$ is real analytic near 0 by Lemma (1.3), there exists $C > 0$ such that

$$|a^{(j)}(0)/j!| \le C^{j+1}, \quad j = 0, 1, 2, \ldots. \tag{1.11}$$

We shall first show

$$a^{(j)}(0)/j! = \frac{1}{\pi}\int_{\mathbb{R}}\frac{f(y)}{y^{j+1}}dy, \quad j = 0, 1, 2, \ldots. \tag{1.12}$$

For this, we begin with (1.4) with $n = 0$ and successively use (1.10) to obtain for $n = 0, 1, 2, \ldots$,

$$a(x) = \chi(x)g(x) + \frac{1}{\pi}\sum_{j=0}^{n}\left(\int\frac{f(y)}{y^{j+1}}dy\right)x^j - x^{n+1}K(\frac{f(y)}{y^{n+1}})(x). \tag{1.13}$$

Since $g(x)$ vanishes of infinite order at 0, and $K(\frac{f(y)}{y^{n+1}})(x)$ is continuous near 0 by Lemma (1.3), (1.12) follows from (1.13).

Recall, by the assumption of the theorem, that f is nonnegative. We now claim that f vanishes identically in an open interval containing 0, which will imply the desired conclusion of the theorem. For this, assume by contradiction that for every positive ϵ we have

$$\int_{-\epsilon}^{\epsilon} f(y)dy > 0. \tag{1.14}$$

From (1.11) and (1.12) we obtain for every j odd,

$$C^{j+1} \ge \frac{1}{\pi}\int_{-\epsilon}^{\epsilon}\frac{f(y)}{y^{j+1}}dy \ge \frac{1}{\pi\epsilon^{j+1}}\int_{-\epsilon}^{\epsilon} f(y)dy, \tag{1.15}$$

which implies

$$\epsilon C \ge \left(\frac{1}{\pi}\int_{-\epsilon}^{\epsilon} f(y)dy\right)^{1/(j+1)}. \tag{1.16}$$

Using (1.14) and letting j go to infinity, we obtain $C \geq 1/\epsilon$. Since ϵ is an arbitrary positive number, we reach a contradiction. The proof of Theorem 1 is now complete. ♠

REMARK (1.17). Note that if in Theorem 1, (ii) is replaced by the stronger condition $\Re h(z) \geq 0$ for $z \in \overline{\Omega}^+$, then the conclusion of the theorem follows from the classical Hopf Lemma.

§2. Applications of Theorem 1.

In this section we shall give a number of applications of Theorem 1 to unique continuation of flat functions in other settings.

THEOREM 2. *Let $\Omega \subset \mathbf{C}$ be a connected open neighborhood of 0. Suppose that $H : \overline{\Omega}^+ \to \mathbf{C}^n$ is continuous, holomorphic in Ω^+, and vanishes of infinite order at 0, i.e. its components satisfy (i) of Theorem 1. If H satisfies*

$$H(\Omega \cap \mathbf{R}) \subset \{z \in \mathbf{C}^n : |\Im z| \leq \beta |\Re z|\} \tag{2.1}$$

for some β with $0 \leq \beta < 1$, then $H \equiv 0$. If $n = 1$ the conclusion holds for $\beta = 1$ also.

PROOF. We write $H = (H_1, \ldots, H_n)$ with $H_j = u_j + iv_j$, where the u_j and v_j are real. By the hypothesis (2.1), we have

$$\sum_j v_j(x)^2 \leq \beta^2 \sum_j u_j(x)^2, \quad x \in \Omega \cap \mathbf{R}. \tag{2.2}$$

Then the function $h = H_1^2 + \ldots H_n^2$ satisfies the hypotheses of Theorem 1, since $\Re h = \sum_j u_j(x)^2 - \sum_j v_j(x)^2 \geq 0$. We may then conclude $h \equiv 0$, which implies in particular, $\sum_j u_j(x)^2 = \sum_j v_j(x)^2$. Combining this with (2.2) and noting that $\beta < 1$, we obtain $\sum_j u_j(x)^2 = 0$, which implies $u_j(x) = v_j(x) = 0$ for all j, i.e. $H \equiv 0$.

If $n = 1$ and $\beta = 1$, then we note that $H^2 = u^2 - v^2 + 2iuv$ satisfies the hypothesis of Theorem 1, and we conclude immediately that $H \equiv 0$. ♠

REMARK (2.3). The conclusion of Theorem 2 need not hold for $n > 1$ and $\beta = 1$. Indeed, let $H_1 : \overline{\Omega}^+ \to \mathbf{C}$ be a continuous function, holomorphic in Ω^+, vanishing of infinite order at 0, but with $H_1 \not\equiv 0$. Then $H = (H_1, iH_1)$, valued in $\mathbf{C}^2$, satisfies (2.1) with $\beta = 1$, but $H \not\equiv 0$.

COROLLARY (2.4). *If H is as in Theorem 2, but with (2.1) replaced by*

$$H(\Omega \cap \mathbf{R}) \subset M, \tag{2.5}$$

where M is a C^1 totally real submanifold of $\mathbf{C}^n$, then $H \equiv 0$.

PROOF. As in [2, Lemma (1.1)], we may find local holomorphic coordinates $z = (z', z'') \in \mathbf{C}^r \times \mathbf{C}^{n-r}$ such that M is given near 0 by

$$\Im z' = \phi(\Re z'), \quad z'' = \psi(\Re z'), \tag{2.6}$$

with ϕ and ψ of class C^1 defined near 0 in $\mathbf{R}^r$, with $\phi(0) = 0$, $\psi(0) = 0$, $d\phi(0) = 0$, $d\psi(0) = 0$. It then follows immediately that near 0, M is contained in a set of the form (2.1) with $\beta < 1$, hence the corollary. ♠

The following corollary is immediate, since any C^1 curve in $\mathbf{C}^n$ is a totally real manifold.

COROLLARY (2.7). *If H is as in Theorem 2, but with (2.1) replaced by $H(\Omega \cap \mathbf{R})$ contained in a C^1 curve of $\mathbf{C}^n$, then $H \equiv 0$.*

This corollary is generalized in Theorem 3 below to the case where a single C^1 curve in $\mathbf{C}^n$ is replaced by a finite union of such curves.

For the next application, we introduce a class of subsets of the complex plane. Let α be a real number and R the ray defined by $R = \{z \in \mathbf{C} : z = re^{2\pi i\alpha},\ 0 \le r < \infty\}$. A set will be called a *single ray concentrated around* R if it is of the form

$$E_R = \{z \in \mathbf{C} : z = re^{2\pi i\theta},\ |\theta - \alpha| \le \omega(r)\}, \tag{2.8}$$

where $\omega(r)$ is a nonegative function defined for $r \ge 0$ and such that $\lim_{r\to 0} \omega(r) = 0$. A subset E of $\mathbf{C}$ is called *ray concentrated* around the rays $R_1, \ldots, R_k$ if

$$E = \bigcup_{j=1}^{k} E_{R_j}, \tag{2.9}$$

where E_{R_j} is a single ray concentrated set around R_j. In particular, a C^1 curve passing through 0 is ray concentrated around its tangent line at 0 and, more generally, a finite union of such curves forms a ray concentrated set. The reader can also easily check that if q is a positive integer and $H(z) = z^q$ then the image under H of a ray concentrated set is also a ray concentrated set. We give now the main property of these sets of interest to us.

LEMMA (2.10). *Let E be a ray concentrated set given by (2.9). There exists a positive integer q and $\epsilon > 0$ such that if $H(z) = z^q$, then $H(E) \cap \{|z| < \epsilon\}$ is contained in the half space $\{\Re z \ge 0\}$.*

PROOF. By the classical Dirichlet theorem (see e.g. [5, Theorem 201]) for every set of real numbers $\alpha_1, \ldots, \alpha_k$ and every positive integer ν there exist a positive integer q and integers $p_1, \ldots, p_k$ such that

$$|q\alpha_j - p_j| < 1/\nu. \tag{2.11}$$

Now, for $j = 1, \ldots, k$, let $2\pi\alpha_j$ be the angle associated to R_j by (2.8), and choose $\nu > 4$. If q satisfies (2.11) and $H(z) = z^q$, then all the rays $R_j^q = H(R_j)$ lie in the set $\{\Re z > 0\} \cup \{0\}$. Since $H(E)$ is a ray concentrated set around $R_1^q, \ldots, R_k^q$, the lemma follows by taking $\epsilon > 0$ sufficiently small. ♠

With the above notions, we have the following.

THEOREM 3. *Let h be a continuous function in $\overline{\Omega}^+$, holomorphic in Ω^+ and vanishing of infinite order at 0. If*

$$h(\Omega \cap \mathbf{R}) \subset E, \tag{2.12}$$

where E is a ray concentrated set, then $h \equiv 0$. In particular, the conclusion holds if E is a finite union of C^1 curves through 0.

PROOF. Let q be the positive integer given by Lemma (2.10). After shrinking Ω around 0 if necessary, it follows from the lemma that the function h^q satisfies the assumptions of Theorem 1, hence the desired conclusion. ♠

3. A Schwarz reflection principle for real analytic sets.

By a *real analytic set* in $\mathbf{C}$ we mean the locus of zeroes of a real valued real analytic function defined in an open set in $\mathbf{C}$. We have the following results.

THEOREM 4. *Let h be a continuous function in $\overline{\Omega}^+$, holomorphic in Ω^+, and vanishing of infinite order at 0. If $h(\Omega \cap \mathbf{R})$ is contained in a real analytic subset of $\mathbf{C}$, then $h \equiv 0$.*

THEOREM 5. *Let h be a smooth function in $\overline{\Omega}^+$, holomorphic in Ω^+, mapping $\Omega \cap \mathbf{R}$ into a real analytic subset of $\mathbf{C}$; then h extends holomorphically to an open neighborhood of $\overline{\Omega}^+$ in $\mathbf{C}$.*

REMARK (3.1). The conclusion of Theorem 5 need not hold if h is only assumed to be of class C^k, even if k is taken to be large, as shown by the following example. Let k be any positive integer and $h(z) = z^{k+1/2}$ where the square root is defined to be holomorphic in the upper half plane. Then $h(\mathbb{R}) \subset \mathcal{A}$ where $\mathcal{A}$ is given by $xy = 0$. Clearly h does not extend holomorphically to any neighborhood of 0 and h is of class C^k in the half plane $\Im z \geq 0$.

PROOF OF THEOREM 4. We shall show that a real analytic set is a ray concentrated set as defined in §2 (see (2.9)). Theorem 4 will then follow immediately from Theorem 3.

Let $\mathcal{A}$ be a real analytic set. By the Weierstrass Preparation Theorem, and after a complex linear change of coordinates in $\mathbb{C}$, we can assume that near the origin $\mathcal{A}$ is given by

$$y^N + \sum_{j=0}^{N-1} a_j(x) y^j = 0, \tag{3.2}$$

where $z = x + iy$ and the a_j are real valued, real analytic functions vanishing at 0. By the classical Puiseux expansion of the roots of (3.2), we conclude that there exist at most $2N$ curves $\gamma_j : [0,1] \to \mathbb{R}^2$ of class C^1 with $\gamma_j(0) = 0$ and $\gamma_j'(0) \neq 0$ such that for ϵ sufficiently small

$$\mathcal{A} \cap \{|z| < \epsilon\} = \bigcup_{j=1}^{k} \gamma_j([0,1]). \tag{3.3}$$

If we take $R_j = \{(t\gamma_j'(0); t \geq 0)\}$, the set $\mathcal{A}$ is then ray concentrated around $R_1, \ldots, R_k$.♠

PROOF OF THEOREM 5. It suffices to show that h extends holomorphically to a full neighborhood of 0 in $\mathbb{C}$. By Theorem 4, if h vanishes of infinite order at 0, then $h \equiv 0$ and the above conclusion follows trivially. Hence we may assume that $\frac{\partial^n h}{\partial x^n}(0) \neq 0$, where n is minimal with this property. We claim that

$$h(z) = z^n a(z), \tag{3.4}$$

where $a(z)$ is smooth in $\overline{\Omega^+}$, holomorphic in Ω^+ and $a(0) \neq 0$. Indeed, the holomorphic function $h(z)/z^n$ in Ω^+ has a boundary value distribution in $\Omega \cap \mathbb{R}$ since $|h(z)/z^n| \leq C(\Im z)^{-n}$ in Ω^+ (see e.g. [4]). On the other hand $h(x)/x^n$ is smooth in $\Omega \cap \mathbb{R}$ by the Taylor expansion of $h(x)$, and the choice of n. By the regularity of the Poisson transform (see e.g. [8]), it follows that $h(z)/z^n$ is smooth in $\overline{\Omega^+}$, and its boundary value is $h(x)/x^n$. This proves the claim (3.4).

After shrinking Ω around 0, if necessary, we let $f(z) = z[a(z)]^{1/n}$, where the nth root is chosen so that $a(z)^{1/n}$ is smooth in $\overline{\Omega^+}$ and holomorphic in Ω^+. Then $f^n \equiv h$, and hence by the assumption of the theorem, $[f^n(\Omega \cap \mathbb{R})] \subset \mathcal{A}$, where $\mathcal{A}$ is an analytic set, i.e.

$\mathcal{A}$ is given near 0 by $\mathcal{A} = \{g(x,y) = 0\}$, with $g(x,y)$ a real valued, real analytic function in a neighborhood of 0. If we let $\mathcal{B}$ be the real analytic set given by $g(\Re z^n, \Im z^n) = 0$, then, after possibly shrinking Ω, we have

$$f(\Omega \cap \mathbb{R}) \subset \mathcal{B}. \tag{3.5}$$

Since $f'(0) \neq 0$, there is an open interval $I \subset \Omega \cap \mathbb{R}$, $0 \in I$, such that $f(I)$ is a smooth curve which is contained in $\mathcal{B}$ by (3.5). We claim that since $\mathcal{B}$ is a real analytic set, $f(I)$ must be a real analytic (smooth) curve. Indeed, the claim follows from a theorem of Malgrange [7], since both $\mathcal{B}$ and $f(I)$ are of real dimension 1. (Note that this can also be seen by writing the equation for $\mathcal{B}$ in the form (3.2) and using the Puiseux expansion for the roots.) The classical Schwarz reflection principle then implies that f extends holomorphically to a full neighborhood of 0 in $\mathbb{C}$. Hence the same holds for h, which proves Theorem 5.♠

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