Vol. 61, No. 2 DUKE MATHEMATICAL JOURNAL © October 1990

UNIQUE CONTINUATION AND REGULARITY AT THE BOUNDARY FOR HOLOMORPHIC FUNCTIONS

SERGE ALINHAC, M. S. BAOUENDI*, AND LINDA PREISS ROTHSCHILD*

§0. Introduction and main results. The function $f(z) = \exp(-1/z^{1/3})$ is holomorphic in the upper half-plane, smooth in its closure, and vanishes of infinite order at the origin. We shall show (Theorem 1) that if a function h has these properties but also maps an interval containing the origin into a nonsingular C^2 curve, then h must vanish identically. When the curve is real analytic this reduces to the Schwarz reflection principle. We state our first result for a vector valued function.

We recall that if M is a submanifold of $\mathbb{C}^n$ of class C^1, we say that M is *totally real* if $T_m \cap JT_m = \{0\}$ for all $m \in M$, where T_m is the tangent space of M at m and J is the usual multiplication by $\sqrt{-1}$.

We define the Lipschitz space $\Lambda_\alpha(\mathbb{R}^n)$, $\alpha > 0$, as in, e.g., Stein [12]. In particular, $f \in \Lambda_1(\mathbb{R}^n)$ if $f \in L^\infty(\mathbb{R}^n)$ and there is a constant A such that $\|f(x+t) + f(x-t) - 2f(x)\|_\infty \leq A|t|$. Similarly $f \in \Lambda_k(\mathbb{R}^n)$, k a positive integer greater than 1, if $\partial f/\partial x_j \in \Lambda_{k-1}(\mathbb{R}^n)$. For α nonintegral, $\Lambda_\alpha(\mathbb{R}^n)$ is the usual Hölder space. A similar definition can be given for $\Lambda_\alpha(F)$, where F is a closed set of $\mathbb{R}^n$ with sufficiently smooth boundary.

Our first theorem gives both regularity and unique continuation results.

THEOREM 1. *Let Ω be an open neighborhood of 0 in $\mathbb{C}$, $\Omega^+ = \Omega \cap \{w = s + it: t > 0\}$, and M' a totally real manifold of $\mathbb{C}^n$ of class C^k, $k \geq 2$, with $0 \in M'$. If $h: \overline{\Omega^+} \to \mathbb{C}^n$ is continuous and holomorphic in Ω^+ and maps $\overline{\Omega^+} \cap \mathbb{R}$ into M' then $h \in \Lambda_k(\overline{\Omega'^+})$ for every open Ω' relatively compact in Ω. Furthermore, if h vanishes of infinite order at the origin, i.e., $h(w) = O(|w|^N)$ for every N, then h vanishes identically in the connected component of the origin in $\overline{\Omega^+}$.*

If M is a totally real manifold of class C^2, $M \subset \mathbb{C}^p$ of real dimension p, $0 \in M$, then M is given locally (see Lemma 1.1) by

$$M = \{w \in \mathbb{C}^p: \operatorname{Im} w = \varphi(\operatorname{Re} w)\} \tag{0.1}$$

for some $\varphi \in C^2(U)$, U a neighborhood of the origin in $\mathbb{R}^p$, φ real valued with $\varphi(0) = \varphi'(0) = 0$. *A wedge $\mathscr{W}$ of edge M* is then defined as a set of the form

$$\mathscr{W} = \{w \in \mathcal{O}: \operatorname{Im} w - \varphi(\operatorname{Re} w) \in \Gamma\}, \tag{0.2}$$

Received October 3, 1989.
*Supported by NSF Grant DMS 8901268.

where $\mathcal{O}$ is an open neighborhood of 0 in $\mathbb{C}^p$ and $\Gamma \subset \mathbb{R}^p$ is a convex open cone. The following, which is the main result of this paper, is a generalization of Theorem 1.

THEOREM 2. *Let $\mathcal{W}$ be a wedge of $\mathbb{C}^p$ given by* (0.2) *with edge M of class C^2, and $h\colon \bar{\mathcal{W}} \to \mathbb{C}^n$ continuous, holomorphic in $\mathcal{W}$ and satisfying*

(i) *h flat at 0, i.e., for every N there exists C_N such that $|h(w)| \leqslant C_N |w|^N$ for $w \in \mathcal{W}$,*
(ii) *$h(\mathcal{O} \cap M) \subset M'$, with M' a totally real submanifold of $\mathbb{C}^n$ of class C^2.*

Then h vanishes identically in the connected component of 0 in $\bar{\mathcal{W}}$.

It follows from the work of Coupet [5] and Pinchuk and Khasanov [9] that if h satisfies (ii) in Theorem 2 above, then h is in $\Lambda_{1+\alpha}$ up to the edge for all α, $0 < \alpha < 1$. Our Theorem 4 in §4 improves this result by showing $h \in \Lambda_2$. For previous results on regularity see also Chirka [4] and Rosay [10].

The proof of Theorem 2 uses analytic disks with part of their boundary sticking to a totally real manifold. Our construction is adapted to the proof of unique continuation and differs from other such constructions in the literature such as that of Pinchuk [8], Sadullaev [11], Chirka [4], and Coupet [5].

Unique continuation is reduced to Carleman estimates for elliptic inequalities (see Aronszajn [1] and Hörmander [6]). We believe that our Theorem 1 is new even in the case of a scalar function, i.e., $n = 1$ in the statement of Theorem 1. After these results were proved, Bell and Lempert [2] gave a somewhat different approach to some of our results.

The paper is organized as follows. In §1 we prove the regularity part of Theorem 1, and in §2 the unique continuation part. Section 3 is devoted to the construction of analytic discs in wedges, which is used to reduce the proof of Theorem 2 to that of Theorem 1 in §4.

Remarks. 1. If M and M' are real analytic, then the conclusion of Theorem 2 is an immediate consequence of the Edge-of-the-Wedge Theorem. Indeed, M and M' can then be holomorphically flattened to $\mathbb{R}^p$ and $\mathbb{R}^n$ respectively (see Lemma 1.1 below) and h extended holomorphically. For $\bar{z} \in \mathcal{W}$ define $\tilde{h}(z) = \overline{h(\bar{z})}$. Since $\tilde{h}(z)$ is real for z real, we can apply the Edge-of-the-Wedge Theorem to conclude that h extends holomorphically to a full neighborhood of 0; hence, $h \equiv 0$.

2. In Theorem 1, if M' is real analytic then any component of h is real analytic; therefore, if it is flat at 0 it must vanish identically. This is not true if M' is only assumed smooth, as can be shown by the following example. Let $f(w)$ be any nonvanishing smooth function defined in $\{w \in \mathbb{C}, \operatorname{Im} w \geqslant 0\}$, holomorphic for $\operatorname{Im} w > 0$ and flat at 0. Write $f(s) = a(s) + ib(s)$ for $s \in \mathbb{R}$. Let $M' \subset \mathbb{C}^2$ be given by

$$M' = \{(z_1, z_2) \in \mathbb{C}^2, \operatorname{Im} z_1 = 0, \operatorname{Im} z_2 = b(\operatorname{Re} z_1)\}.$$

Let $h(w) = (w, f(w))$; we have $h(\mathbb{R}) \subset M'$, the second component of h is flat without vanishing identically.

§1. Regularity of functions with boundary values in a totally real manifold. In this section we prove the regularity part of Theorem 1. We begin with the following lemma whose proof is elementary and is left to the reader.

LEMMA 1.1. *Let $M \subset \mathbb{C}^n$ be a manifold of class C^k, $k \geq 1$. Then M is totally real if and only if near every $m_0 \in M$ there exist holomorphic coordinates z in $\mathbb{C}^n$ vanishing at m_0 such that, near m_0,*

$$(1.2) \qquad M = \{(z', z'') \in \mathbb{C}^r \times \mathbb{C}^{n-r} : \operatorname{Im} z' = \varphi(\operatorname{Re} z'), z'' = \psi(\operatorname{Re} z')\}$$

with φ, ψ of class C^k defined near the origin in $\mathbb{R}^r$, φ real, with $\varphi(0) = 0$, $\psi(0) = 0$, $d\varphi(0) = 0$, $d\psi(0) = 0$. In addition, if M is real analytic, then after a holomorphic change of coordinates one can take $\varphi \equiv 0$ and $\psi \equiv 0$ in (1.2).

Note that if M is totally real of maximal dimension then $r = n$ in (1.2). As a consequence of Lemma 1.1 it suffices to prove Theorem 1 when M' is totally real of maximal dimension. Indeed, we may write with the notation of Lemma 1.1, $h = (h', h'')$ with h' valued in $\mathbb{C}^r$ and h'' valued in $\mathbb{C}^{n-r}$. With the notation as in (1.2), the submanifold M_1' of $\mathbb{C}^r$ defined by $\{z' \in \mathbb{C}^r : \operatorname{Im} z' = \varphi(\operatorname{Re} z')\}$ is totally real of maximal dimension.

Now assume that Theorem 1 is proved for the case of a totally real manifold of maximal dimension. For the unique continuation part in the general case we conclude that $h' \equiv 0$. Since for $s \in \mathbb{R}$

$$(1.3) \qquad h''(s) = \psi(\operatorname{Re} h'(s)),$$

it follows that $h''(s) \equiv 0$ on $\mathbb{R}$, and hence $h'' \equiv 0$. A similar argument will give the regularity part of Theorem 1. We show first that $h' \in \Lambda_k(\overline{\Omega'^+})$ and hence by (1.3) $h''|_{\mathbb{R}} \in \Lambda_k$; therefore, $h'' \in \Lambda_k(\overline{\Omega'^+})$ by the holomorphy of h'' (See, e.g., [12].)

For the remainder of this section we assume M' totally real of maximal dimension given by

$$(1.4) \qquad \operatorname{Im} z = \varphi(\operatorname{Re} z), \qquad \varphi(0) = 0, \qquad \varphi'(0) = 0.$$

A main point in the proof of the regularity of h is the following distance estimate due to Rosay [10], which generalizes the estimates of Chirka [4] in the case $k \geq 2$. We give here a somewhat simpler proof.

PROPOSITION 1.5. *If h is as in the first statement of Theorem 1 then for $w \in \Omega^+$*

$$(1.6) \qquad \operatorname{dist}(h(w), M') \leq C(\operatorname{Im} w),$$

with C a positive constant, where dist(,) *denotes the Euclidean distance in $\mathbb{C}^n$.*

Proof. For $s+it \in \Omega^+$ we write $u(s,t) = \operatorname{Re} h(s+it)$, $v(s,t) = \operatorname{Im} h(s+it)$, and define a real vector valued function $q(s,t)$ by

$$q(s,t) = v(s,t) - \varphi(u(s,t)), \tag{1.7}$$

where φ is given in (1.4). If $q = (q_1, \ldots, q_n)$, $\varphi = (\varphi_1, \ldots, \varphi_n)$, and $\Delta = \partial^2/\partial s^2 + \partial^2/\partial t^2$ we have, since $\Delta u_j = \Delta v_j = 0$,

$$\Delta q_j = -\sum_{k,l} \varphi_{j,x_k x_l}(u)(u_{k,s}u_{l,s} + u_{k,t}u_{l,t}), \tag{1.8}$$

where $(z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n)$. If the Hessian of (φ_j) is positive definite we obtain from (1.8) that $\Delta q_j \leqslant 0$. Since $q_j(s,0) = 0$ by assumption, we conclude by the maximum principle that $q_j(s,t) \geqslant -Ct$ for some $C > 0$. We now need the following.

LEMMA 1.9. *If $M' \subset \mathbb{C}^n$ is a totally real submanifold of maximal dimension of class C^k, $k \geqslant 2$, and $0 \in M'$, there exist holomorphic coordinates $z = x + iy \in \mathbb{C}^n$ near 0 such that M' is defined by*

$$y = \varphi(x), \quad \textit{with } \varphi(0) = 0, \quad \varphi'(0) = 0, \quad \varphi''(0) > 0 \quad (\textit{resp.} < 0), \tag{1.10}$$

where $\varphi \in C^k$ and $\varphi''(0)$ denotes the Hessian.

Proof. By Lemma 1.1 we may assume that M' is defined by $y' = \tilde{\varphi}(x')$, with $z' = x' + iy'$ being the coordinates in $\mathbb{C}^n$, with $\tilde{\varphi}(0) = 0$ and $\tilde{\varphi}'(0) = 0$. We make the change of coordinates

$$z_j = z_j' + iC \sum_{k=1}^{n} z_k'^2.$$

Then M' is defined by $y = \varphi(x)$, $\varphi = (\varphi_1, \ldots, \varphi_n)$, $\varphi_j(x) = \tilde{\varphi}_j(\theta(x)) + C[\sum(\theta_k(x))^2 - (\tilde{\varphi}_k(\theta(x)))^2]$, where $\theta(x)$ is the inverse map of $x = (x_j) \mapsto (x_j - 2C\sum x_k \tilde{\varphi}_k(x))$. Since $\theta(x) = x + O(|x|^2)$, the lemma follows by taking C large enough.

We continue the proof of Proposition 1.5. By Lemma 1.9 we obtain defining functions φ and $\hat{\varphi}$ for M', with respect to different sets of coordinates, such that the Hessian of φ is positive definite, and the Hessian of $\hat{\varphi}$ is negative definite. Let q and $\hat{q}$ be the corresponding vector-valued real functions defined by (1.7). Then for $j = 1, \ldots, n$, we have, by the maximum principle, as explained above

$$q_j(s,t) \geqslant -Ct \quad \text{and} \quad \hat{q}_j(s,t) \leqslant Ct. \tag{1.11}$$

Also, by the construction of φ and $\hat{\varphi}$ we have

$$q(s,t) = A(s,t)\hat{q}(s,t), \tag{1.12}$$

where $A(s, t)$ is a matrix with continuous coefficients and $A(0) = I_{n,n}$. We claim that there exists C' such that

$$(1.13)\qquad |q_j(s, t)| \leqslant C't, \qquad |\hat{q}_j(s, t)| \leqslant C't.$$

To prove (1.13) we note first that by (1.12) we have

$$\sum_{j=1}^{n} q_j = \sum_{j=1}^{n} \alpha_j \hat{q}_j, \qquad \alpha_j > 0,$$

so that by (1.11)

$$(1.14)\qquad -Ct \leqslant \sum_{j=1}^{n} q_j(s, t) \leqslant C''t.$$

Then since $-2q_1 \leqslant Ct$ by (1.11) we have

$$(1.15)\qquad \sum_{j=2}^{n} q_j - q_1 \leqslant C'''t.$$

Adding (1.14) and (1.15) gives

$$(1.16)\qquad -Ct \leqslant \sum_{j=2}^{n} q_j \leqslant Ct,$$

with possibly a different constant C. Proceeding by induction we obtain the claim (1.13).

The inequality (1.6) then follows by noting that $\operatorname{dist}(h(w), M') \sim \sum_{j=1}^{n} |q_j(s, t)|$. The proofs of Lemma 1.9 and hence of Proposition 1.5 are now complete.

We shall need the following routine result which gives an almost holomorphic extension of a function of class C^k, $k \geqslant 1$.

LEMMA 1.17. *Let $f(x)$ be a function of class C^k, $k \geqslant 1$, defined in a neighborhood of 0 in $\mathbb{R}^n$. Then there exists $\tilde{f}(x, y)$ of class C^k defined in a neighborhood of 0 in $\mathbb{R}^{2n}$ satisfying*

$$(1.18)\qquad \tilde{f}(x, 0) = f(x) \quad \text{and} \quad |\bar{\partial}\tilde{f}(x, y)| = O(|y|^{k-1}),$$

with $\bar{\partial} = (\partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_n})$, where $\partial_{\bar{z}_j} = \frac{1}{2}((\partial/\partial x_j) + i(\partial/\partial y_l))$.

Proof. This can be proved by writing

$$(1.20)\qquad \tilde{f}(x, y) = \sum_{0 \leqslant |\beta| \leqslant k} \lambda_\beta f(x + (\beta)y)$$

where $\beta = (\beta_1, \ldots, \beta_n)$, with β_j a nonnegative integer, $y = (y_1, \ldots, y_n)$, $|\beta| = \sum_{j=1}^{n} \beta_j$ and $(\beta)y = (\beta_1 y_1, \beta_2 y_2, \ldots, \beta_n y_n)$, $\lambda_\beta \leqslant \mathbb{C}$. Requiring that for every multi-index γ, $|\gamma| \leqslant k$,

$$\partial_y^\gamma \tilde{f}(x, 0) = i^{|\gamma|} \partial_x^\gamma f(x), \tag{1.21}$$

leads to the system

$$\sum_{0 \leqslant |\beta| \leqslant k} \beta^\gamma \lambda_\beta = i^{|\gamma|}, \qquad |\gamma| \leqslant k,$$

with $\beta^\gamma = \beta_1^{\gamma_1} \ldots \beta_n^{\gamma_n}$, $\gamma = (\gamma_1, \ldots, \gamma_n)$, which has a unique solution in the unknowns λ_β. The proof of Lemma 1.17 follows from (1.21).

PROPOSITION 1.22. *Let $M' \subset \mathbb{C}^n$ be a totally real manifold of class C^k, $k \geqslant 1$, of maximal dimension. Then there exists a C^k diffeomorphism $z = F(z', \bar{z}')$ defined near the origin in $\mathbb{C}^n$ such that F maps a neighborhood of 0 in $\{\operatorname{Im} z' = 0\}$ into M' and $|\bar{\partial} F(z', \bar{z}')| = O(|\operatorname{Im} z'|^{k-1})$. Also F^{-1} satisfies $|\bar{\partial} F^{-1}(z, \bar{z})| = O(\operatorname{dist}(z, M')^{k-1})$.*

Proof. We assume M' is defined by (1.4), and let $\tilde{\varphi}(x', y')$ be the vector-valued function associated to φ by Lemma 1.17. It suffices to take $F(z', \bar{z}') = z' + i\tilde{\varphi}(x', y')$ to prove the Proposition.

In order to prove the regularity claimed in Theorem 1 we shall need the following result.

PROPOSITION 1.23. *Let f be in $C^0(\overline{\Omega^+}) \cap C^1(\Omega^+)$, where Ω^+ is as in Theorem 1. In addition, we assume that f is real valued on $\overline{\Omega^+} \cap \{t = 0\}$. If there exist $C > 0$ and $\sigma > -1$ such that*

$$|\bar{\partial} f(s, t)| \leqslant C t^\sigma, \tag{1.24}$$

then the function $s \mapsto f(s, 0)$ is in $\Lambda_{1+\sigma}$.

Proof. Defining $\tilde{f}(s, t)$ by

$$\tilde{f}(s, t) = \begin{cases} f(s, t) & t > 0 \\ \overline{f(s, -t)} & t < 0 \end{cases} \tag{1.25}$$

we have

$$|\bar{\partial} \tilde{f}(s, t)| \leqslant C |t|^\sigma \tag{1.26}$$

in an open neighborhood of 0. Note that $\tilde{f}$ is C^1 away from $t = 0$. Using the generalized Cauchy integral formula we have, for $s \in \mathbb{R}$, $|s|$ small,

$$f(s) = \frac{1}{2\pi i} \left[\int_{\partial \Omega'} \frac{\tilde{f}(s', t')}{w' - s} \, dw' + \iint_{\Omega'} \frac{\bar{\partial} \tilde{f}(s', t')}{w' - s} \, dw' \wedge d\bar{w}' \right] \tag{1.27}$$

where Ω' is an open neighborhood of 0 in $\mathbb{C}$. We shall restrict ourselves to the case where σ is an integer k; the other cases may be done similarly, but are easier to handle. From (1.27) we have $f(s) = p(s) + v(s)$, where

$$v(s) = \iint_{\Omega'} \frac{g(s', t')}{w' - s} \, ds' \, dt' \quad \text{with} \quad |g(s', t')| \leqslant C|t'|^k, \tag{1.28}$$

and $p(s)$ real analytic near 0. By induction, it suffices to show $v^{(k)}(s) \in \Lambda^1$, i.e.,

$$v^{(k)}(s + \delta) + v^{(k)}(s - \delta) - 2v^{(k)}(s) = O(|\delta|). \tag{1.29}$$

From (1.28), the left hand side of (1.29) is

$$k! \iint g(s', t') \left[\frac{1}{(w' - s - \delta)^{k+1}} + \frac{1}{(w' - s + \delta)^{k+1}} - \frac{2}{(w' - s)^{k+1}} \right] ds' \, dt'. \tag{1.30}$$

We put $X = w' - s - \delta$, $Y = w' - s + \delta$ and write the rational function in (1.30) over the common denominator $X^{k+1} Y^{k+1} (X + Y)^{k+1}$. The numerator is a sum of terms of the form $\delta^j X^{k+1-j} Y^{k+1}$, $\delta^j Y^{k+1-j} X^{k+1}$, $2 \leqslant j \leqslant k$, and $\delta Y^k X^k [Y - X] = 2\delta^2 Y^k X^k$.

We estimate these terms separately. First consider a term of the form

$$\iint_{\Omega'} \frac{g(s', t') \delta^j Y^{k+1-j} X^{k+1}}{X^{k+1} Y^{k+1} (X + Y)^{k+1}} \, ds' \, dt' = \iint_{\Omega'} \frac{\delta^j g(s', t')}{(w' - s + \delta)^j (w' - s)^{k+1}} \, ds' \, dt', \tag{1.31}$$

where $2 \leqslant j \leqslant k$. To estimate the right-hand side of (1.31) we subdivide Ω' into regions Ω'_1 and Ω'_2 given by

$$\Omega'_1 = \{w' \in \Omega' : |w' - s + \delta| > |w' - s|\}, \qquad \Omega'_2 = \{w' : |w' - s + \delta| \leqslant |w' - s|\}.$$

To estimate the integral over Ω'_1 we denote by B_1 the disc of radius $\delta/2$ centered at $s - \delta$, i.e., $B_1 = \{w' : |w' - s + \delta| < (\delta/2)\}$. We introduce polar coordinates $w' - s + \delta = re^{i\theta}$. In B_1 we use the inequality $|w' - s| > \delta/2$, and in $\Omega'_1 \backslash B_1$ the inequality $|w' - s| > r$. Using (1.28) we have $|g(s', t')| \leqslant C|t'|^k \leqslant Cr^k$. We have

$$\left| \iint_{B_1} \frac{\delta^j g}{(w' - s + \delta)^j (w' - s)^{k+1}} \, ds' \, dt' \right| \leqslant C \int_0^{|\delta|/2} \frac{|\delta|^j r^k}{r^j |\delta|^{k+1}} r \, dr \leqslant C' |\delta|.$$

Similarly,

$$\left| \iint_{\Omega'_1 \backslash B_1} \frac{\delta^j g}{(w' - s + \delta)^j (w' - s)^{k+1}} \, ds' \, dt' \right| \leqslant C \int_{|\delta|/2}^{R} \frac{|\delta|^j r^k}{r^{j+k+1}} r \, dr \leqslant C' |\delta|.$$

Similar estimates hold for the integral over Ω'_2.

It remains to consider

(1.32)
$$\iint_{\Omega'} \frac{g\delta^2 Y^k X^k \, ds'\, dt'}{X^{k+1} Y^{k+1} (X+Y)^{k+1}} = \iint_{\Omega'} \frac{g\delta^2 \, ds'\, dt'}{[(w'-s-\delta)(w'-s+\delta)(w'-s)]^{k+1}}.$$

For this we write $\Omega' = \Omega_1' \cup \Omega_2' \cup \Omega_3'$ with

$$\Omega_1' = \{w' \in \Omega_1' : |w'-s+\delta| \leqslant \min(|w'-s|, |w'-s-\delta|)\}$$

$$\Omega_2' = \{w' \in \Omega' : |w'-s| \leqslant \min(|w'-s+\delta|, |w'-s-\delta'|)\}$$

$$\Omega_3' = \{w' \in \Omega' : |w'-s-\delta| \leqslant \min(|w'-s|, |w'-s+\delta|)\}.$$

As before we write $\Omega_1' = (\Omega_1' \backslash B_1) \cup B_1$ and in Ω_1' introduce polar coordinates $w' - s + \delta = re^{i\theta}$. In B_1 we use the inequalities $|w'-s-\delta| > (\delta/2)$ and $|w'-s| > (\delta/2)$ and in $\Omega_1' \backslash B_1$ we use $|w'-s-\delta| > r$ and $|w'-s| > r$. We use similar estimates in Ω_2' and Ω_3' with polar coordinates centered at s and $s+\delta$ respectively. Each integral over Ω_j', $j = 1, 2, 3$ is bounded by $C'|\delta|$. This completes the proof of Proposition 1.23.

We now prove the regularity of the function h of Theorem 1. We give the proof only for $k = 2$; the general case may be done by iterating the argument. Let $G = F^{-1}$, where $F: \mathbb{C}^n \to \mathbb{C}^n$ is a local C^2 diffeomorphism as in Proposition 1.22. Put $f = G \circ h$. Then f maps $\overline{\Omega^+} \cap \mathbb{R}$ into $\mathbb{R}^n$, and

$$\partial_{\bar{w}} f = G_{\bar{z}}(h) \partial_{\bar{w}} \bar{h} + G_z(h) \partial_{\bar{w}} h = G_{\bar{z}}(h) \overline{\partial_w h}, \tag{1.33}$$

since $\partial_{\bar{w}} h \equiv 0$. By Proposition 1.22 and Proposition 1.5 we have

$$|G_{\bar{z}}(h(w))| \leqslant C \operatorname{dist}(h(w), M) \leqslant C't, \tag{1.34}$$

where $w = s + it$. Also, since h is continuous in $\overline{\Omega^+}$, by the Cauchy integral formula we have

$$|\partial_w h(w)| \leqslant Ct^{-1}. \tag{1.35}$$

Combining (1.33), (1.34) and (1.35) we obtain

$$|\partial_{\bar{w}} f| \leqslant C. \tag{1.36}$$

By Proposition 1.23 we have $f(s, 0) \in \Lambda_1$. Since $f = G \circ h$ and G is a diffeomorphism of class C^2 we conclude that $s \mapsto h(s)$ is also in $\Lambda_1 \subset \Lambda_{1/2}$. We claim that

$$|\partial_w h(s+it)| \leqslant C|t|^{-1/2}. \tag{1.37}$$

Indeed, if $u(s) = \chi(s)h(s)$ with $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ near 0, and P_t is the Poisson operator

$$P_t u(s) = \frac{1}{\pi} \int \frac{tu(s-s')\,ds'}{t^2 + (s-s')^2},$$

then since $u \in \Lambda_{1/2}(\mathbb{R})$ the inequality $|\partial_t P_t u(s)| \leqslant C|t|^{-1/2}$ holds (see, e.g., [12]). Now, since $h(s+it) - P_t u(s)$ is harmonic and vanishes for $t = 0$, we conclude it is real analytic for $t \geqslant 0$ in a neighborhood of 0, from which (1.37) follows.

Using (1.37) and (1.34) in (1.33) we obtain

$$(1.38) \qquad |\partial_{\bar{w}} f| \leqslant Ct^{1/2}.$$

Therefore by Proposition 1.23, $f(s, 0) \in \Lambda_{3/2} \subset C^1$, which implies that $s \mapsto h(s) \in C^1$. Since h is holomorphic, $h(s+it)$ is of class C^1 for $t \geqslant 0$ near the origin which implies $\partial_w h$ bounded. Using (1.33) and (1.34) again we conclude that $|\partial_{\bar{w}} f| \leqslant Ct$, which implies that $s \mapsto h(s)$ is locally in $\Lambda_2(\mathbb{R})$. Since $h(s+it)$ is harmonic, the regularity of the Dirichlet problem in Lipschitz spaces [12] implies the desired regularity result of Theorem 1 for $k = 2$.

§2. Proof of unique continuation in Theorem 1. Let h be as in the hypotheses of Theorem 1. By the regularity result proved in §1, we can assume that $h \in \Lambda_2(\overline{\Omega^+})$. Let q be defined by (1.7). We shall first prove the claim that if h is as in Theorem 1 (flat at 0) then

$$(2.1) \qquad q(s, t) \equiv 0.$$

Using the Cauchy-Riemann equations we have, for $j = 1, \ldots, n$,

$$(2.2) \qquad q_{j,s} = u_{j,t} - \sum_k \varphi_{j,x_k}(u) u_{k,s}, \qquad q_{j,t} = -u_{j,s} - \sum_k \varphi_{j,x_k}(u) u_{k,t}.$$

Equations (2.2) form a system of $2n$ linear equations for the $2n$ unknowns $u_{j,t}$ and $u_{j,s}$. Since $d\varphi(0) = 0$ we obtain

$$(2.3) \qquad u_{j,t} = \sum_k (a_{jk} q_{k,t} + b_{jk} q_{k,s}), \qquad u_{j,s} = \sum_k (c_{jk} q_{k,t} + d_{jk} q_{k,s}),$$

where the coefficients $a_{jk}, b_{jk}, c_{jk}, d_{jk}$ are continuous functions in a neighborhood of 0 in $\overline{\Omega^+}$. By using (1.8) and (2.3) we obtain the following estimates for $j = 1, \ldots, n$:

$$(2.4) \qquad |\Delta q_j(s, t)| \leqslant C \sum_{k=1}^{n} (|q_{k,s}(s, t)| + |q_{k,t}(s, t)|),$$

for $|s| < \varepsilon$, $0 < t < \varepsilon$, with $\varepsilon > 0$.

We now extend q_j for $t < 0$ by defining

$$\tilde{q}_j(s, t) = \begin{cases} q_j(s, t) & 0 < t < \varepsilon \\ -q_j(s, -t) & -\varepsilon < t < 0 \end{cases}. \tag{2.5}$$

Note that by (1.7) and the assumption that h maps the real line into M' we have $q_j(s, 0) \equiv 0, j = 1, \ldots, n$. It follows that $\tilde{q}_j$ defined by (2.5) is of class C^1 for $|s| < \varepsilon$, $|t| < \varepsilon$. By the flatness assumption on h at the origin we have that for each N there exists C_N such that

$$|\tilde{q}_j(s, t)| \leqslant C_N(|s| + |t|)^N. \tag{2.6}$$

Since $\tilde{q}_j$ is of class C^2 away from $t = 0$ we have $\Delta\tilde{q}_j = \widetilde{\Delta q_j}$. From (2.4) we obtain

$$|\Delta\tilde{q}_j(s, t)| \leqslant C \sum_{k=1}^{n} (|\tilde{q}_{k,s}(s, t)| + |\tilde{q}_{k,t}(s, t)|), \tag{2.7}$$

for $|s| < \varepsilon$, $|t| < \varepsilon$, $t \neq 0$.

We shall use the following classical unique continuation result.

THEOREM A. *Let $f = (f_1, \ldots, f_n)$, $f_j \in L^2(B_\varepsilon)$, where B_ε is the ball in $\mathbb{R}^2$ centered at 0, and of radius ε. Assume that $\Delta f_j \in L^2(B_\varepsilon)$ and the following holds almost everywhere in B_ε:*

$$\sum_{j=1}^{n} |\Delta f_j(x)| \leqslant C \sum_{j=1}^{n} |\operatorname{grad} f_j(x)|.$$

If, in addition, for every N and $j = 1, \ldots, n$,

$$\int_{|x|<\eta} |f_j(x)|\, dx = O(\eta^N),$$

then $f \equiv 0$.

Theorem A is essentially due to Carleman [3]; see also Aronszajn [1], Hörmander [6].

We may apply Theorem A to $\tilde{q}$, since (2.6) and (2.7) hold, to conclude $\tilde{q} \equiv 0$, which proves the claim (2.1). To prove that (2.1) implies the vanishing of h also we use (2.1) and solve for $v(s, t)$ in (1.7) to obtain

$$h(s, t) = u(s, t) + i\varphi(u(s, t)). \tag{2.8}$$

Since h is holomorphic, we may apply $\partial_s + i\partial_t$ to (2.8), which yields $(\partial_s + i\partial_t)u \equiv 0$. The reality of u and the vanishing of u at 0 imply that $u \equiv 0$. Since this implies $v \equiv 0$ also, the unique continuation part of Theorem 1 is proved.

§3. Construction of analytic discs; proof of unique continuation in Theorem 2. Let M be a totally real manifold of class C^2 of maximal dimension contained in $\mathbb{C}^p$. As before, we may assume that M is given by

$$y = \varphi(x), \quad \varphi = (\varphi_1, \ldots, \varphi_p), \quad \varphi_j \in C^2, \quad \partial_x^\alpha \varphi_j(0) = 0, |\alpha| \leqslant 2. \tag{3.1}$$

Let $\mathscr{W}$ be the wedge with edge M defined by (0.2). We shall prove the following.

THEOREM 3. *There exist V an open subset of S^{p-1}, the unit sphere of $\mathbb{R}^p$, and a map χ*

$$\chi\colon D_\varepsilon^+ \times V \to \mathscr{W},$$

where $D_\varepsilon^+ = \{w \in \mathbb{C}\colon |w| < \varepsilon, \operatorname{Im} w > 0\}$, with $\chi \in C^0(\overline{D_\varepsilon^+} \times V)$, $\chi(w, \alpha)$ holomorphic in w for $w \in D_\varepsilon^+$ satisfying

(i) $\chi(0, \alpha) = 0$ *for all* $\alpha \in V$,
(ii) $\chi(I \times V) \subset M$, $I = (-\varepsilon, \varepsilon)$, *and* $\operatorname{interior}_M \chi(I \times V) \neq \varnothing$.

Proof. Let $\rho(t)$ be a C_0^∞ function on the real line satisfying $0 \leqslant \rho(t) \leqslant 1$, $\rho(t) \equiv 0$ for $|t| > 2$, $\rho(t) \equiv 1$ for $|t| \leqslant 1$. If $\eta > 0$ and φ is as in (3.1) we write

$$\varphi_\eta(x) = \rho(|x|/\eta)\varphi(x). \tag{3.2}$$

Let $H = (H^1(-1, 1))^p$, the vector-valued L^2 Sobolev space of order 1 on $(-1, 1)$. We denote by $E = \operatorname{Lip}(S^{p-1}, H)$ the set of functions $f(s, \alpha)$, $s \in (-1, 1)$, $\alpha \in S^{p-1}$ such that $s \mapsto f(s, \alpha)$ is in H, and

$$\|f(\cdot, \alpha) - f(\cdot, \beta)\|_H \leqslant C|\alpha - \beta|$$

for all $\alpha, \beta \in S^{p-1}$. It follows from Ascoli's Theorem and the compactness of the embedding of H in $(L^2(-1, 1))^p$ that the closed unit ball B_1 in E (with the obvious norm) is a compact subset of $C^0(S^{p-1}, (L^2(-1, 1))^p)$.

We let T denote the Hilbert transform on $\mathbb{R}$ defined by

$$Tu(s) = \mathrm{PV}\int_{-\infty}^{\infty} \frac{u(s - s')}{s'}\, ds',$$

and let $T_{(-1,1)}u = (Tu)|_{(-1,1)}$, the restriction to $(-1, 1)$.

LEMMA 3.3. *If r and η are sufficiently small positive numbers, then*

$$g \mapsto T_{(-1,1)}[s^{-1}\varphi_\eta(s(\alpha + g(s, \alpha)))] \tag{3.4}$$

maps B_r, the closed ball of radius r in E, into itself and is continuous for the topology of $C^0(S^{p-1}, L^2)$.

Proof. By the Sobolev embedding theorem, we can choose r sufficiently small so that $\|g\|_H \leqslant r$ implies $\sup_s |g(s)| < \frac{1}{2}$. We now fix r satisfying this condition. If $s > 4\eta$ then $|s(\alpha + g(s))| \geqslant |s|(|\alpha| - |g(s)|) \geqslant 2\eta$. Hence

$$\operatorname{supp}\left(s \mapsto \frac{1}{s}\varphi_\eta(s(\alpha + g(s)))\right) \subset (-4\eta, 4\eta), \tag{3.5}$$

by (3.2). We will require $\eta < \frac{1}{4}$. We note that (3.5) still holds if we only assume $g \in H$ with $\|g\|_H \leqslant r$. This may be seen by extending g to $H^1(\mathbb{R})$ with norm $\leqslant r + \varepsilon$, ε small. For $g \in B_r$, the function in (3.5) will be extended by zero outside the interval $(-1, 1)$. Since $T_{(-1,1)}$ maps $H_0^1(-1, 1)$ into $H^1(-1, 1)$ and $L^2_{\text{comp}}(-1, 1)$ into $L^2(-1, 1)$ with norm 1, it suffices to prove the conclusion of the lemma for the map

$$g \mapsto s^{-1}\varphi_\eta(s(\alpha + g(s, \alpha))). \tag{3.6}$$

We shall first show that η can be chosen, $\eta < \frac{1}{4}$, so that (3.6) maps B_r into itself, with r fixed as above. It suffices to show that if $g(s, \alpha)$ satisfies

$$\sup_{\alpha \in S^{p-1}} \|g(\cdot, \alpha)\|_H + \sup_{\alpha \in S^{p-1}} \|g'_\alpha(\cdot, \alpha)\|_H < r, \tag{3.7}$$

then

$$\sup_{\alpha \in S^{p-1}} \left\|\frac{1}{s}\varphi_\eta(s(\alpha + g(s, \alpha))\right\|_H + \sup_{\alpha \in S^{p-1}} \left\|\partial_\alpha\left(\frac{1}{s}\varphi_\eta(s(\alpha + g(s, \alpha)))\right)\right\|_H < r. \tag{3.8}$$

We use the following estimates for φ, which follow from (3.1) and (3.2):

$$|\varphi_\eta(x)| \leqslant C|x|^2 \leqslant 4C\eta^2, |\varphi'_\eta(x)| \leqslant C|x| \leqslant 2C|\eta|, |\varphi''_\eta(x)| \leqslant C. \tag{3.9}$$

Since $\|f\|_H^2 = \int_{-1}^1 |f(s)|^2\, ds + \int_{-1}^1 |f'(s)|^2\, ds$, each of the norms in (3.8) may be estimated by two $L^2(-1, 1)$ norms. This is done by expanding the derivatives and estimating each resulting L^2 norm separately. We shall indicate here the estimates only for a few of the terms which arise. In estimating

$$\sup_{\alpha \in S^{p-1}} \left\|\partial_s\partial_\alpha\left[\frac{1}{s}\varphi_\eta(s(\alpha + g))\right]\right\|_{L^2(-1,1)}$$

we obtain terms such as

$$\|g_{\alpha s}\varphi'_\eta(s(\alpha + g))\|_{L^2} \leqslant \sup|\varphi'_\eta| \sup_\alpha \|g_{\alpha s}\|_{L^2} \leqslant C|\eta| r.$$

Another term is

$$\sup_{\alpha \in S^{p-1}} \|g_\alpha\varphi''_\eta(s(\alpha + g))\cdot\alpha\|_{L^2} \leqslant \sup|\varphi''_\eta| \sup|g_\alpha|\left(\int_{-2\eta}^{2\eta} ds\right)^{1/2} \leqslant C|\eta|^{1/2},$$

where we use the Sobolev embedding theorem to estimate $\sup |g_\alpha|$.

We now prove the continuity of the map (3.6) in the norm of $F = C^0(S^{p-1}, L^2)$. In fact, we shall prove that the map (3.6) is a contraction in F. For $g_1, g_2 \in F$ we have, by the mean value theorem,

$$\sup_\alpha \left\| \frac{1}{s}\varphi_\eta(s(\alpha + g_1)) - \frac{1}{s}\varphi_\eta(s(\alpha + g_2)) \right\|_{L^2} \leqslant \sup |\varphi_\eta'| \sup_\alpha \|g_1 - g_2\|_{L^2}.$$

By (3.9) we obtain the desired conclusion. This completes the proof of Lemma 3.3.

We may now prove Theorem 3. By Lemma 3.3 and the Brouwer Fixed Point Theorem there exists $g \in B_r \subset E = \operatorname{Lip}(S_\alpha^{p-1}, H)$ such that

$$g = -T_{(-1,1)}[s^{-1}\varphi_\eta(s(\alpha + g(s, \alpha))]. \tag{3.10}$$

We let $\tilde{g}$ be defined by

$$\tilde{g} = -T[s^{-1}\varphi_\eta(s(\alpha + g(s, \alpha)))]. \tag{3.11}$$

Since $\|\tilde{g}(\cdot, \alpha)\|_{H^1(\mathbb{R})} \leqslant r$, we have, by (3.11) and the remarks in the beginning of the proof of Lemma 3.3,

$$\tilde{g} = -T[s^{-1}\varphi_\eta(s(\alpha + \tilde{g}(s, \alpha)))]. \tag{3.12}$$

Applying T to both sides of (3.12) and using the fact that $T^2 = -I$ we obtain

$$T\tilde{g}(s, \alpha) = s^{-1}\varphi_\eta(s(\alpha + \tilde{g}(s, \alpha))). \tag{3.13}$$

For $w \in \mathbb{C}$, $\operatorname{Im} w \geqslant 0$, and $\alpha \in S^{p-1}$, we define

$$\chi(w, \alpha) = w\alpha + w\mathscr{G}(w, \alpha), \tag{3.14}$$

where $w \mapsto \mathscr{G}(w, \alpha)$ is the holomorphic function in the upper half space defined by

$$\mathscr{G}|_{t=0} = \tilde{g} + iT\tilde{g} \tag{3.15}$$

with $\tilde{g}$ as in (3.13). It follows from (3.13), (3.14) and (3.15) that we have

$$\operatorname{Im} \chi(s, \alpha) = \varphi_\eta(\operatorname{Re} \chi(s, \alpha)), \tag{3.16}$$

for $s \in \mathbb{R}$ and $\alpha \in S^{p-1}$. We need to find V open in $S^{p-1} \cap \Gamma$ (where Γ is as in (0.2)) so that for $\alpha \in V$ and $w \in \mathbb{C}$, $|w| < \varepsilon$, $\operatorname{Im} w > 0$ we have

$$\chi(w, \alpha) \in \mathscr{W}, \qquad \text{i.e., } \operatorname{Im} \chi(w, \alpha) - \varphi_\eta(\operatorname{Re} \chi(w, \alpha)) \in \Gamma. \tag{3.17}$$

Since

(3.18)
$$\operatorname{Im}\chi(w,\alpha) = t\alpha + t\operatorname{Re}\mathscr{G} + s\operatorname{Im}\mathscr{G} \text{ and } \varphi_\eta(\operatorname{Re}\chi(w,\alpha)) = \varphi_\eta(s\alpha + s\operatorname{Re}\mathscr{G} - t\operatorname{Im}\mathscr{G}),$$

taking $\alpha_0 \in S^{p-1} \cap \Gamma$ and V an open neighborhood of α_0 properly contained in $S^{p-1} \cap \Gamma$, we see from (3.16) and (3.18) that (3.17) holds if r and ε are sufficiently small.

It remains to verify condition (ii) of Theorem 3. Since g and $\tilde{g}$ agree in a neighborhood of the origin in s, we shall regard g as defined in $\mathbb{R} \times S^{p-1}$; therefore (3.12) is satisfied with $\tilde{g}$ replaced by g. To simplify notation we shall write φ for φ_η in (3.12). We shall first prove further regularity for g.

LEMMA 3.19. *Let g be a solution of* (3.12). *Then the components of g_α and sg_s are in $C^0(S^{p-1}, H^1(\mathbb{R}))$ with norms $\leqslant C(r)$, with r as in Lemma* 3.3, *and* $\lim_{r\to 0} C(r) = 0$.

Proof. We first prove regularity of g_α. Differentiating (3.12) we obtain for $j = 1, \ldots, n-1$,

$$g_{\alpha_j} = -T[s^{-1}\varphi'(s(\alpha + g(s,\alpha)))\cdot((s\cdot\alpha)_{\alpha_j} + sg_{\alpha_j})], \tag{3.20}$$

where $(\alpha_1, \ldots, \alpha_{p-1})$ are local coordinates on S^{p-1}. From (3.20) we may write

$$g_{\alpha_j} = a + T(bg_{\alpha_j}), \tag{3.21}$$

where $a = -T[s^{-1}\varphi'(s(\alpha + g(s,\alpha)))((s\cdot\alpha)_{\alpha_j})]$ and $b = -\varphi'(s(\alpha + g(s,\alpha)))$. Since $g \in E$, we have $g_\alpha \in L^\infty(S^{p-1}, H^1)$. We shall show that $g_\alpha \in C^0(S^{p-1}, H^1)$. It is easy to check by (3.9) and the properties of g that a and b are in $C^0(S^{p-1}, H^1)$, with norms which go to 0 as $r \to 0$. This shows, since T is bounded on H^1, that the map

$$u \mapsto a + T(bu)$$

is a contraction in both $C^0(S^{p-1}, H^1)$ and $L^\infty(S^{p-1}, H^1)$. The desired regularity of g_{α_j} follows.

To prove the regularity of sg_s we first observe that we have the following commutation relation:

$$T(sv)_s = (sTv)_s, \tag{3.22}$$

for all $v \in L^2(\mathbb{R})$ for which $sv_s \in L^2(\mathbb{R})$. Using (3.22) and differentiating (3.12) we obtain

$$(sg)_s = -T[\varphi'(s(\alpha + g))(\alpha + (sg)_s)]. \tag{3.23}$$

We observe first that $(sg)_s \in C^0(S^{p-1}, L^2)$, by noting the fact that $\varphi'(s(\alpha + g))$ has compact support and that $g_s \in L^2$, so that the right-hand side of (3.23) is in $C^0(S^{p-1}, L^2)$. As before we may write (3.23) in the form

$$v = c + T(dv),$$

with $v = (sg)_s$, where $c = -T[\varphi'(s(\alpha + g))\alpha]$ and $d = -\varphi'(s(\alpha + g))$. It is easy to see that c and d are both in $C^0(S^{p-1}, H^1)$ with norms going to 0 as $r \to 0$. A contraction argument in this space shows $sg_s \in C^0(S^{p-1}, H^1)$ also. This proves Lemma (3.19).

To prove (ii) of Theorem 3 it suffices to take r and η small and use the implicit function theorem in conjunction with the following lemma.

LEMMA 3.24. *There exists a positive constant C_p, depending only on p, such that if $\Theta: \mathbb{R} \times S^{p-1} \to \mathbb{R}^p$ is of the form*

$$\Theta(s, \alpha) = s\alpha + sg(s, \alpha)$$

with $g, g_\alpha, sg_s \in C^0(\mathbb{R} \times S^{p-1})$ and $\|g\| + \|g_\alpha\| + \|sg_s\| \leqslant C_p$, where $\|\ \|$ is the sup norm in $\mathbb{R} \times S^{p-1}$, then $J(\Theta)(s, \alpha) \neq 0$ for $s \neq 0$, $\alpha \in S^{p-1}$, where $J(\Theta)$ denotes the Jacobian determinant of Θ.

Proof. It suffices to choose coordinates on S^{p-1} and to compute the Jacobian determinant. After factoring out s in the $p - 1$ columns in the matrix corresponding to the differentiations in α, the result follows from the assumptions of small norms of $\|g\|$, $\|g_\alpha\|$, and $\|sg_s\|$.

We may now prove Theorem 2. Let $\chi_{(\alpha)}(w) = \chi(w, \alpha)$, where χ is given by Theorem 3 and $\alpha \in V$. If h satisfies the assumption of Theorem 2, then the function $h \circ \chi_{(\alpha)}$ satisfies the assumptions of Theorem 1. Hence $h \circ \chi_{(\alpha)} \equiv 0$ on $\chi_{(\alpha)}(I)$, with $I = (-\varepsilon, \varepsilon)$. Letting α vary in V and using (ii) of Theorem 3, we conclude that $h \equiv 0$ on an open set of the totally real manifold M. It follows from a well known result that this implies $h \equiv 0$ in $\mathcal{W}$.

§4. Boundary regularity in the general case. In this section we prove the analogue of the regularity result of Theorem 1. This generalizes the results of Coupet [5] and Pinchuk and Khasanov [9] to the case of the spaces Λ_k, for integral k.

THEOREM 4. *Let $\mathcal{W}$ be a wedge of $\mathbb{C}^p$ given by (0.2) with edge M and $h: \bar{\mathcal{W}} \to \mathbb{C}^n$ continuous and holomorphic in $\mathcal{W}$ such that $h(\mathcal{O} \cap M) \subset M'$, with M' a totally real submanifold of $\mathbb{C}^n$. If M and M' are of class C^k, $k \geqslant 2$, then the restriction of h to M is in $\Lambda_k(M)$.*

Proof. We shall give the proof only in the case $k = 2$. For $k > 2$ the proof follows by iterating the argument. By Lemma 1.1 we may assume that M' is of maximum dimension in $\mathbb{C}^n$. By the regularity results in [5] and [9], we may assume that $h \in C^1(\bar{\mathcal{W}})$. (However, this could have been proved directly by the techniques of §1.) Let F be the mapping given by Proposition 1.22 and set $G = F^{-1}$. Similarly, we denote by $\mathscr{F}$ the map corresponding to F for M' replaced by M. Then $\mathscr{F}$ maps a flat wedge $\mathscr{V}$ of $\mathbb{C}^p$ with edge an open neighborhood of 0 in $\mathbb{R}^p$ to $\mathscr{W}$. After shrinking Γ we can assume that $\mathscr{V}$ is given by

$$\mathscr{V} = \{\zeta \in \mathscr{U} : \operatorname{Im} \zeta \in \Gamma\}, \tag{4.1}$$

where $\mathscr{U}$ is an open neighborhood of 0 in $\mathbb{C}^p$. If $\alpha \in \Gamma \cap S^{p-1}$ and $\beta \in \mathbb{R}^p$ we define the analytic disc

$$\chi(w) = \chi_{\alpha,\beta}(w) = w\alpha + \beta. \tag{4.2}$$

For w small, $\operatorname{Im} w > 0$, we have $\chi(w) \in \mathscr{V}$. We put

$$\psi = h \circ \mathscr{F} \circ \chi \quad \text{and} \quad f = G \circ \psi. \tag{4.3}$$

We first compute $\bar{\partial}\psi$. We have

$$\bar{\partial}\psi(w, \bar{w}) = \partial_{\bar{w}}(h(\mathscr{F}(\chi(w), \overline{\chi(w)}))) = h'(\mathscr{F}(\chi(w), \overline{\chi(w)})) \cdot \mathscr{F}_{\bar{\zeta}}(\chi(w), \overline{\chi(w)}) \cdot \overline{\chi'(w)}. \tag{4.4}$$

Similarly,

$$\partial\psi(w, \bar{w}) = \partial_w(h(\mathscr{F}(\chi(w), \overline{\chi(w)}))) = h'(\mathscr{F}(\chi(w), \overline{\chi(w)})) \cdot \mathscr{F}_\zeta(\chi(w), \overline{\chi(w)}) \cdot \chi'(w). \tag{4.5}$$

Using Proposition 1.22 to estimate $\mathscr{F}_{\bar{\zeta}}$ and the fact that h' and χ' are bounded, we conclude

$$|\bar{\partial}\psi(w, \bar{w})| \leqslant C|\operatorname{Im} w|. \tag{4.6}$$

From (4.5) we obtain only that $|\partial\psi(w)|$ is bounded by a constant. Using these, we now estimate $\bar{\partial}f$. We have

$$\begin{aligned} \partial_{\bar{w}} f = {} & \partial_{z'} G(\psi(w, \bar{w}), \overline{\psi(w, \bar{w})}) \partial_{\bar{w}} \psi(w, \bar{w}) \\ & + \partial_{\bar{z}'} G(\psi(w, \bar{w}), \overline{\psi(w, \bar{w})}) \cdot \overline{\partial_w \psi(w, \bar{w})}. \end{aligned} \tag{4.7}$$

Using Proposition 1.22 to estimate $G_{\bar{z}'}$ and (4.6), we have

$$|\bar{\partial}f(w, \bar{w})| \leqslant C|\operatorname{Im} w|. \tag{4.8}$$

Since f maps $\mathbb{R}$ into $\mathbb{R}^n$ we may apply Proposition 1.24 to conclude that $s \mapsto f(s, 0)$ is in Λ_2.

Since $\mathscr{F}$ and G are C^2 local diffeomorphisms, the proof of Theorem 4 will be completed by noting that the constant in (4.8) is uniform in α and β, and hence, so is the Λ_2 norm of $f(s, 0)$, and by using the following lemma.

LEMMA 4.9. *For $\alpha \in S^{p-1}$ and $\beta \in \mathbb{R}^p$ define $\mu_{\alpha,\beta}: t \in \mathbb{R} \mapsto t\alpha + \beta \in \mathbb{R}^p$. Suppose that $f: \mathbb{R}^p \mapsto \mathbb{R}$ so that $f \circ \mu_{\alpha_j,\beta} \in \Lambda_k(\mathbb{R})$ for all β and for $\alpha_1, \ldots, \alpha_p$ a linearly independent set of vectors with $\|f \circ \mu_{\alpha_j,\beta}\|_{\Lambda_k(\mathbb{R})} \leqslant C$, with C independent of β and j; then $f \in \Lambda_k(\mathbb{R}^p)$.*

Proof. We shall use the characterization of $\Lambda_k(\mathbb{R}^p)$ in terms of the Littlewood-Paley decomposition (see, e.g., [7]), which we recall here briefly. Let $\psi \in C_0^\infty(\mathbb{R}^p)$, $\psi(\xi) = 1$ for $|\xi| \leqslant 1/2$, $\psi(\xi) = 0$ for $|\xi| \geqslant 1$. Set $\varphi(\xi) = \psi(\xi/2) - \psi(\xi)$; then φ is supported in the ring $1/2 \leqslant |\xi| \leqslant 2$, and for every ξ,

$$1 = \psi(\xi) + \sum_{q \geqslant 0} \varphi(2^{-q}\xi).$$

For u a tempered distribution in $\mathbb{R}^p$ we set $u_{-1} = S_0 u = \psi(D)u$, $u_q = \varphi(2^{-q}D)u$, so that $u = S_0 u + \sum_{q \geqslant 0} u_q$. This is the Littlewood-Paley decomposition of u. The space $\Lambda_k(\mathbb{R}^p)$ is then characterized by the property

$$\|u_q\|_{L^\infty} \leqslant C 2^{-kq}, \tag{4.10}$$

the best constant C being equivalent to the norm in $\Lambda_k(\mathbb{R}^p)$.

Considering, instead of f, the function $g(x) = f(\sum x_j \alpha_j)$, we can always assume the directions $\alpha_1, \ldots, \alpha_p$ to be $(1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)$. To take advantage of the product structure in $\mathbb{R}^p$, we use now equivalently, instead of the balls and rings defined above an appropriate sequence of nested dyadic cubes.

In order to prove (4.10) it is enough to check the boundedness of the pieces of f corresponding to any face of any dyadic cube, for instance, a piece of the form

$$Qf = \tilde{\psi}(2^{-l}D')\varphi(2^{-l}D_p)f \tag{4.11}$$

with $\tilde{\psi}(\xi') = \psi(\xi_1)\ldots\psi(\xi_{p-1})$, where φ and ψ are functions in one variable chosen as above.

Now set $h = \varphi(2^{-l}D_p)f$. For any $x' \in \mathbb{R}^{p-1}$, it follows from the hypotheses and (4.10) (in one variable) that

$$\|h(x', \cdot)\|_{L^\infty(\mathbb{R}_{x_p})} \leqslant C 2^{-kl} \|f(x', \cdot)\|_{\Lambda_k(\mathbb{R})} \leqslant C' 2^{-kl},$$

and hence $\|h\|_{L^\infty(\mathbb{R}^p)} \leqslant C' 2^{-kl}$. For any $x_p \in \mathbb{R}$ it follows from (4.11) that

$$\|Qf(\cdot, x_p)\|_{L^\infty(\mathbb{R}^{p-1}_{x'})} \leqslant C'' \|h(\cdot, x_p)\|_{L^\infty(\mathbb{R}^{p-1}_{x'})},$$

since the L^1 norm of the inverse Fourier transform of $\tilde{\psi}(2^{-l}\xi')$ is independent of l. Thus with a different constant C we have $\|Qf\|_{L^\infty(\mathbb{R}^p)} \leqslant C 2^{-kl}$ which proves $f \in \Lambda_k(\mathbb{R}^p)$ and completes the proof of the lemma.

§5. Further results and remarks. Our proof uses the assumptions that both M and M' are of class C^2. It would be interesting to know whether unique continuation holds in Theorems 1 and 2 under weaker assumptions, even in the case $n = p = 1$.

We address now a somewhat different question of unique continuation. If S is a real, smooth hypersurface in $\mathbb{C}^p$ defined by $\rho(z, \bar{z}) = 0$, $\rho(0) = 0$, $d\rho(0) \neq 0$, and Ω is an open neighborhood of 0 in $\mathbb{C}^p$, we write $\Omega^+ = \{z \in \Omega, \rho(z, \bar{z}) > 0\}$. Let h: $\Omega^+ \to \mathbb{C}^n$ be a holomorphic mapping, $h \in C^\infty(\overline{\Omega^+})$, vanishing of infinite order at 0, and $h(S) \subset S'$, where S' is another hypersurface of $\mathbb{C}^n$. What conditions should be imposed on S and S' to insure that $h \equiv 0$?

As an example, consider the case where S and S' are hypersurfaces in $\mathbb{C}^3$ and S' is given by $\operatorname{Im} w = |z_1|^2 - |z_2|^2$. The mapping $h = (f, f, 0)$, where f is holomorphic in Ω^+, $C^\infty(\overline{\Omega^+})$ and flat at 0, does not necessarily vanish identically. It is not known to the authors whether or not unique continuation holds, for example, when S and S' are real analytic hypersurfaces of $\mathbb{C}^n$ and S' contains no germ of a complex variety containing zero. However, we have the following immediate corollary of Theorem 2.

COROLLARY. *Let S be a real hypersurface of class C^2 contained in $\mathbb{C}^p$ given by $\rho(z, \bar{z}) = 0$, $\rho(0) = 0$, $d\rho(0) \neq 0$. Let Ω be a small neighborhood of 0 in $\mathbb{C}^p$, $\Omega^+ = \{z \in \Omega, \rho(z, \bar{z}) > 0\}$, and h: $\Omega^+ \to \mathbb{C}^n$ holomorphic, $h \in C^2(\overline{\Omega^+})$. Assume*

(i) *h vanishes of infinite order at 0, i.e., $|h(z)| \leqslant C_N |z|^N$ for $z \in \Omega^+$.*
(ii) *There exists a totally real manifold M of dimension p of class C^2 such that $0 \in M \subset S$, $h(M) \subset M'$, where M' is a totally real submanifold of $\mathbb{C}^n$ of class C^2.*

Then $h \equiv 0$ in the component of 0 in $\overline{\Omega^+}$.

REFERENCES

1. N. ARONSZAJN, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pures Appl. 36 (1957), 235–249.
2. S. BELL AND L. LEMPERT, *A C^∞ Schwarz reflection principle in one and several complex variables* (preprint).
3. T. CARLEMAN, *Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendentes*, Arkiv Mat. Astr. Fys. 26B, no. 17, (1939), 1–9.
4. E. M. CHIRKA, *Regularity of the boundaries of analytic sets*, Math. USSR Sbornik 45, No. 3 (1983), 291–335.
5. B. COUPET, *Régularité d'applications holomorphes sur des variétés totalement réelles*, Thesis, University of Provence, France (1987).
6. L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators, I*, Springer, Berlin-Heidelberg-New York, 1983.
7. Y. MEYER, *Remarques sur un théorème de J. M. Bony*, Rendiconti Circ. Mat. Palermo, Suppl. no. 1 (1981), 1–20.
8. S. I. PINCHUK, *A boundary uniqueness theorem for holomorphic functions of several complex variables*, Math. Notes 14 (1974), 116–120.
9. S. I. PINCHUK AND S. V. KHASANOV, *Asymptotically holomorphic functions and their applications*, Math. USSR Sbornik 62, No. 2 (1989), 541–550.
10. J.-P. ROSAY, *Une remarque à propos de fonctions holomorphes et du front d'onde C^∞*, Comm. P.D.E. 12 (1987), 1323–1332.
11. A. SADULLAEV, *A boundary uniqueness theorem in $\mathbb{C}^n$*, Math. USSR Sbornik 101, No. 4 (1976), 501–514.

12. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ (1970).

Alinhac: Université de Paris-Sud, Département de Mathématique, 91405 Orsay, France
Baouendi and Rothschild: Department of Mathematics, University of California-San Diego, La Jolla, California 92037