Homework 5 Solutions to Selected Problems

February 25, 2012

1 Chapter 5, Problem 2c (not graded)

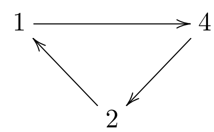
We are given the permutation

$$(12)(13)(23)(142)$$

and need to (re)write it as a product of disjoint cycles. It helps to write out the permutation in array form, and then determine the disjoint cycles.

To determine the array form, we need to figure out what the permutation does to the numbers 1, 2, 3, and 4. Let us see where 1 goes. The problem is that as we go from right to left, each cycle sends 1 elsewhere, and we need to track this new number as we move to the left:

- (142) corresponds to the permutation

  so it sends 1 to 4. Now we need to track 4 as we move to the left.

- (23) fixes 4 since 4 does not appear in this cycle.
- (13) fixes 4
- (12) fixes 4

Therefore, (12)(13)(23)(142) takes 1 to 4.

Now let us see where 2 goes:

- (142) takes 2 to 1
- (23) fixes 1

- (13) corresponds to

$$\begin{tikzcd} 1 \arrow[r, bend left] & 3 \arrow[l, bend left] \end{tikzcd}$$

  so it takes 1 to 3

- (12) fixes 3

Therefore, (12)(13)(23)(142) takes 2 to 3.

Where does 3 go?

- (142) fixes 3
- (23) takes 3 to 2
- (13) fixes 2
- (12) takes 2 to 1

Therefore, (12)(13)(23)(142) takes 3 to 1.

Finally, we determine what the permutations does to 4.

- (142) takes 4 to 2
- (23) takes 2 to 3
- (13) takes 3 to 1
- (12) takes 1 to 2

(12)(13)(23)(142) takes 4 to 2. Thus, the permutation in array form is

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 1 & 2 \end{bmatrix}$$

To convert this to disjoint cycle form, start with 1, and look where it goes (to 4). Then check where 4 goes, then where that number goes, until you return to 1. The number that goes to 1 ends the cycle. After that, look for a number that does not appear in the cycle, and repeat. It may help to draw more arrows in the array that move upwards from a number in the second row to where it appears in the top row:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{bmatrix}$$

we see that 1 goes to 4, which goes to 2, which goes to 3, which goes back to 1:

$$(1423)$$

Since there are no unused numbers, we are done. Hence

$$(12)(13)(23)(142) = (1423).$$

That is, $(12)(13)(23)(142)$ and (1423) are the same permutation, just written in different ways.

2 Chapter 5, Problem 6 (graded)

We want a permutation in A_8 which has order 15. Thus, we will have to:

1. Determine which disjoint cycle structures correspond to permutations in S_8 with order 15 by looking at the least common multiple of the lengths of the cycles. The sum of the lengths of the cycles cannot be more than 8, since we want the permutation to be in S_8, the permutations on the set of eight elements $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

2. Determine whether the permutations with the disjoint cycles structures from part 1 are even permutations (and hence are in A_8).

To get a permutation with order 15 in S_8, we need a five cycle multiplied with a three cycle:

$$(\underline{5})(\underline{3})$$

An example (there are others) of a permutation of this form is

$$(12345)(678)$$

This is because the only way to pick two numbers whose least common multiple is 15 and whose sum is less than or equal to 8 is to choose 5 and 3.

Next, we need to determine if such a permutation is even or odd. That is, if we rewrote $(12345)(678)$ as a product of 2-cycles, will we have an even number or an odd number of 2-cycles? If we look at the proof of Theorem 5.4, we see that a cycle of length k can be written as a product of $k-1$ 2-cycles. In general, if you have a cycle

$$(a_1 a_2 \dots a_{k-1} a_k)$$

we can rewrite it as a product of two cycles

$$(a_1 a_k)(a_1 a_{k-1}) \dots (a_1 a_2).$$

Thus,

$$(12345) = (15)(14)(13)(12)$$

can be written as 4 2-cycles, and

$$(678) = (68)(67)$$

can be written as two 2-cycles. Hence their product can be written using six $(4+2=6)$ 2-cycles. We can write it by replacing each of the original cycles by their product of 2-cycles. In this example, I replace (678) by (68)(67) and then I replace (12345) by (15)(14)(13)(12):

$$(12345)(678) = (12345)(68)(67)$$

$$= (15)(14)(13)(12)(68)(67)$$

Hence (12345)(678) is an even permutation with order 15.

3 Chapter 5, Problem 18b (graded)

3.1 Without finding $\alpha\beta$ in disjoint cycle form

Let us write α and β as disjoint cycles first, and then convert them into products of 2-cycles. We can then multiply the 2-cycles to express $\alpha\beta$ as a product of 2-cycles without having to compute $\alpha\beta$ as a product of disjoint cycles.

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix}$$

Thus, α is a product of two disjoint cycles:

$$\alpha = (12345)(678).$$

Convert to 2-cycle form:

$$\alpha = (15)(14)(13)(12)(68)(67).$$

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$$

1 goes to itself, so we get a cycle of (1) by itself.

$$\beta = (1)(23847)(56) = (23847)(56)$$

In general, if you have a permutation that is a product of more than one cycles, and one of its cycles has only one number in it, you can omit that cycle.

$$\beta = (27)(24)(28)(23)(56)$$

Hence we can write $\alpha\beta$ by multiplying the 2-cycles:

$$\alpha\beta = (15)(14)(13)(12)(68)(67)(27)(24)(28)(23)(56).$$

That is, we write the 2-cycles for α first, followed by the 2-cycles for β.

3.2 Finding $\alpha\beta$ in disjoint cycle form first

We can use the array forms for α and β to find the array form of $\alpha\beta$. First, remember that multiplication of permutations must be read from right to left, so to determine what $\alpha\beta$ does, we perform β first and then α. To write $\alpha\beta$, we can create an array with three rows.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \end{bmatrix}$$

The first row will have the numbers from 1 to 8 in order. The second row will be the second row of the array for β. The third row is different: to find it, it helps if you take the array for α and reorder the columns so that the top row matches the second row of β:

$$\alpha = \begin{bmatrix} 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \\ 2 & 4 & 6 & 8 & 7 & 1 & 3 & 5 \end{bmatrix}$$

The second row of this array for α with be the third row for the large array:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \\ 2 & 4 & 6 & 8 & 7 & 1 & 3 & 5 \end{bmatrix}$$

Then the array for $\alpha\beta$ will be the first and third rows of this array:

$$\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 7 & 1 & 3 & 5 \end{bmatrix}$$

Thus, we can write $\alpha\beta$ as a product of disjoint cycles:

$$\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 7 & 1 & 3 & 5 \end{bmatrix}$$

$$\alpha\beta = (12485736).$$

Then we can write $\alpha\beta$ as a product of 2-cycles:

$$\alpha\beta = (16)(13)(17)(15)(18)(14)(12).$$

4 Chapter 5, Problem 26 (not graded)

Note that 3-cycles are even permutations - a 3-cycle of the form $(a_1a_2a_3) = (a_1a_3)(a_1a_2)$. Hence any product of 3-cycles is a product of even permutations, and hence is even as well. However, (1234) is an odd permutation. We can write it as a product of three 2-cycles:

$$(1234) = (14)(13)(12)$$

It is not possible for an odd permutation to be a product of even permutations.

5 Chapter 5, Problem 28 (graded)

We are given a permutation $\beta = (123)(145)$ which is not in disjoint cycle form, and we need to compute a huge power of it. Perhaps we can find the order of β to help reduce the work. How will this help? Say β has order n. That is, $\beta^n = \varepsilon$, the identity permutation that does not move any numbers. Then we can use the division algorithm on 99 and n:

$$99 = qn + r$$

where $0 \leq r < n$ (see Theorem 4.1, page 73). Thus,

$$\beta^{99} = \beta^{qn+r} = (\beta^n)^q\beta^r = \varepsilon^q\beta^r = \varepsilon\beta^r = \beta^r.$$

How do we determine the order of β? The best way to do this is to convert β into a product of disjoint cycles, and take the least common multiple of the lengths of the cycles. First, we convert to array form:

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 1 & 5 & 2 \end{bmatrix}$$

$$\beta = (14523)$$

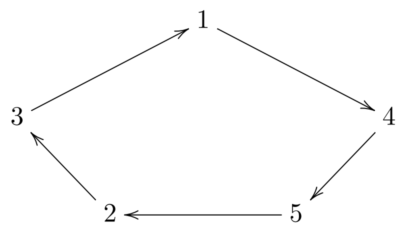
Thus, the order of β is 5. Hence $\beta^5 = \varepsilon$, so

$$\beta^{99} = \beta^{5\cdot 19+4} = (\beta^5)^{19}\beta^4 = \varepsilon^{19}\beta^4 = \varepsilon\beta^4 = \beta^4.$$

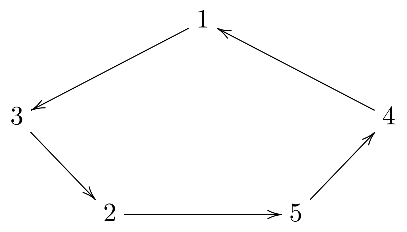
$$\beta^4 = (14523)(14523)(14523)(14523) = (13254)$$

There must be an easier way to find β^4. Fortunately, since $4 = -1 \bmod 5$, $\beta^4 = \beta^{-1}$ (see Theorem 4.1), so instead of multiplying β four times, we only need to compute its inverse.

First, we can write β using a circle:



Remember that β^{-1} should "undo" β. Hence in the diagram, all we need to do is make the arrows point in the opposite direction:



$$\beta^{-1} = (13254).$$

6 Chapter 5, Problem 43 (not graded)

Note: I will only look at elements of order 2.

A_5 consists of even permutations (that is, permutations which can be written as a product of an even number of 2-cycles). The ones with order 2 will consist of a product of disjoint cycles for which the least common multiple of the lengths of the cycles is 2. Hence, we want permutations which satisfy the following:

1. The sum of the lengths of the cycles cannot be more than 5 (so that the permutation is in S_5).
2. Each permutation has a disjoint cycle structure consisting of products of 2-cycles only (to get an order of 2).
3. Each permutation can be written as a product of an even number of 2-cycles (so that the permutation is even and in A_5).

Hence, our permutations must have the form

$$(\underline{2})(\underline{2})$$

How do we count them? One way to do it is to "fill in the blanks:"

$$(\underline{??})(\underline{??})$$

We need to fill in the question marks with the numbers 1, 2, 3, 4, and 5 (remember we are working in S_5). We have 5 choices for the first mark, 4 for the second mark, 3 for the third mark, and 2 for the fourth mark. This gives us a preliminary total of

$$5 \cdot 4 \cdot 3 \cdot 2 = 120.$$

However, this method will count permutations like (12)(34) and (21)(34) as different, even though despite the different appearances of the cycles: they both swap 1 and 2, and they both swap 3 and 4. Similarly, the permutations (12)(34) and (34)(12) are counted as different, but they are equal (disjoint cycles commute). How do we avoid double counting? We need to do some division to get rid of the double counting.

In the first cycle, we can rotate the numbers to get the same cycle. This tells us that our method counts two permutations as different if the only difference is that their first cycles are rotated. So we must divide by 2. Similarly, in the second cycle, we can rotate the numbers without changing the cycle. This tells us that our method counts two permutations as different if the only difference is that their second cycles are rotated. Again, we need to divide by 2:

$$\frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 2} = 30.$$

Now we must address the issue of reordering the 2-cycles themselves. Given a pair of disjoint 2-cycles, there are $2! = 2$ ways to reorder them, and we must count them as the same. Thus, we need to divide once more by 2!:

$$\frac{\frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 2}}{2!} = 15.$$

Thus, there are a total of 15 elements in A_5 of the form $(\underline{2})(\underline{2})$, and these are the permutations with order 2.

7 Chapter 6, Problem 4 (graded)

To show that two groups G and H are not isomorphic, we must show that there is NO isomorphism from one to the other. Typically, this takes the form of a proof by contradiction that assumes there is an isomorphism $\phi : G \to H$ (we could also do $\phi : H \to G$). Usually, we will try to write down what ϕ does to certain elements, using the three properties of an isomorphism:

1. ϕ is a one-to-one function.
2. ϕ is an onto function.
3. ϕ preserves the operation: For any two elements a, b in G,

$$\phi(ab) = \phi(a)\phi(b)$$

We often arrive at a contradiction where ϕ fails to satisfy one of the three properties. In addition, we can also use Theorems 6.2 and 6.3 to help obtain a contradiction.

Here are multiple ways to solve this problem:

7.1 Using Theorem 6.3

Now, suppose that $\phi : U(8) \to U(10)$ is an isomorphism. By Theorem 6.3, since $U(10)$ is cyclic, then so is $U(8)$, which is false. Hence, there is no isomorphism from $U(8)$ to $U(10)$.

7.2 Using Theorem 6.2, Property 7

Suppose that $\phi : U(8) \to U(10)$ is an isomorphism. $U(8)$ and $U(10)$ are both finite, so by Theorem 6.2, property 7, $U(8)$ and $U(10)$ have the same number of elements of order 2. But $U(8)$ has three elements of order 2 (3, 5, and 7), while $U(10)$ only has one element of order 2 (9), a contradiction.

7.3 Using Theorem 6.2, Properties 1 and 5

Suppose that $\phi : U(8) \to U(10)$ is an isomorphism. Let us write down what ϕ does to each number in $U(8)$. First, $\phi(1) = 1$, because 1 is the identity element of $U(8)$ and $U(10)$. Now, ϕ is one-to-one and onto, so we know there is only one element $a \in U(8)$ for which $\phi(a) = 9$. We note that $a \neq 1$, and $|a| = 2$ since every element other than 1 has order 2 in $U(8)$. Let b be another element of $U(8)$ not equal to a or 1. Then b has order 2 as well, so $\phi(b)$ must have order 2 in $U(10)$. Since the only element of order 2 in $U(10)$ is 9, we must have $\phi(b) = 9$, contradicting the assumption that ϕ is a one-to-one function.

8 Chapter 6, Problem 5 (not graded)

To show that two groups are isomorphic, you must define a function from one group to the other, and show that your function is one-to-one, onto, and operation-preserving. Theorems 6.2 and 6.3 are useless in this situation!

Here is one way of finding an isomorphism from $U(8)$ to $U(12)$. First, write their Cayley tables:

$U(8)$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

$U(12)$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

We need to define $\phi : U(8) \to U(12)$ so that it is one-to-one, onto, and operation-preserving. We can do this by defining ϕ element by element, applying ϕ to each entry in the table for $U(8)$, and checking if we have the table for $U(12)$.

Let us try:

$$\phi(1) = 1$$
$$\phi(3) = 5$$
$$\phi(5) = 7$$
$$\phi(7) = 11$$

So far, ϕ is one-to-one (no number appears more than once on the right side of each equation) and onto (every number in $U(12)$ appears on the right side). Let us apply ϕ to each entry in the table for $U(8)$:

$\phi(U(8))$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

This is the same as the Cayley table for $U(12)$. This means that our function ϕ preserves the operation in the groups, and therefore is an isomorphism.

9 Chapter 6, Problem 10 (graded)

9.1 α is an automorphism implies G is Abelian

Given any two elements a, b in G, we need to show that $ab = ba$. Since α is an automorphism, we know that

$$\alpha(ab) = \alpha(a)\alpha(b).$$

Let us use the fact that by definition, $\alpha(a) = a^{-1}$ and $\alpha(b) = b^{-1}$:

$$(ab)^{-1} = a^{-1}b^{-1}$$

By Theorem 2.4, $(ab)^{-1} = b^{-1}a^{-1}$:

$$b^{-1}a^{-1} = a^{-1}b^{-1}$$

Invert both sides:

$$ab = ba$$

Therefore, G is Abelian.

9.2 G is Abelian implies α is an automorphism

9.2.1 α is One-to-one

Suppose $\alpha(a) = \alpha(b)$. Then $a^{-1} = b^{-1}$, so by inverting both sides, $a = b$. Hence α is one-to-one.

9.2.2 α is Onto

Let $b \in G$. We need to find an element $a \in G$ so that $\alpha(a) = b$. That is, we need to solve for a in terms of b.

First, if $\alpha(a) = b$, then $a^{-1} = b$. Thus, $a = b^{-1}$, and we have

$$\alpha(b^{-1}) = \left(b^{-1}\right)^{-1} = b.$$

Therefore, α is onto.

9.2.3 α Preserves the Group Operation: $\alpha(ab) = \alpha(a)\alpha(b)$

$$\alpha(ab) = (ab)^{-1} = b^{-1}a^{-1}$$

Since G is Abelian,

$$b^{-1}a^{-1} = a^{-1}b^{-1} = \alpha(a)\alpha(b)$$

Therefore, α preserves the group operation. Hence it is an automorphism.

10 Chapter 6, Problem 14 (not graded)

According to Theorem 6.5, $\mathrm{Aut}(Z_6)$ is isomorphic to $U(6) = \{1, 5\}$. Hence, there are only two automorphisms of Z_6: α_1 and α_5 (the subscripts come from $U(6)$). How does this help us write down what the automorphisms do to a number in Z_6? Since Z_6 is cyclic, all we need to know is where each automorphism takes 1, the generator of Z_6. By Theorem 4.2, property 4, if α is an automorphism of Z_6, $\alpha(1)$ must be another generator of Z_6. The generators of Z_6 are the numbers between 1 and 5 which are coprime to 6, namely 1 and 5 - the same numbers in $U(6)$. Hence, there are two automorphisms on Z_6. The first one, α_1, takes 1 to 1. The second automorphism, α_5, takes 1 to 5.

11 Chapter 6, Problem 20 (not graded)

1 is a generator of Z_{50}, so we need to know what $\phi(1)$ is. Then if n is any integer between 1 and 49, $\phi(n) = \phi(n \cdot 1) = n \cdot \phi(1)$.

First, we are given that $\phi(11) = 13$. We need an integer m so that $m \cdot 11 = 1 \pmod{50}$. Then $\phi(m \cdot 11) = m \cdot \phi(11) = m \cdot 13$.

We can use brute force to list some multiples (positive and negative) of 11:

$$-110, -99, -88, -77, -66, -55, -44, -33, -22, -11, 0, 11, 22, 33, 44$$

We see that $-99 = 1 \pmod{50}$, so we can take $m = -9$. Hence

$$\phi(1) = \phi(-9 \cdot 11) = -9 \cdot \phi(11) = -9 \cdot 13 = 33 \bmod 50$$

$$\phi(1) = 33.$$

Now that we know $\phi(1)$, the rest of the function can be defined as explained at the beginning of this solution.

12 A1

Let α be a 6-cycle for which $\alpha^2 = (123)(456)$. To find α, it helps to start with a circle diagram. We know 1 has to appear in the cycle, since a 6-cycle on $\{1, 2, 3, 4, 5, 6\}$ must use all six numbers:

$$\begin{tikzcd}
 & 1 \arrow[dr] & \\
? \arrow[ur] & & ? \arrow[d] \\
? \arrow[u] & & ? \arrow[dl] \\
 & ? \arrow[ul] &
\end{tikzcd}$$

α^2 takes 1 to 2. On the circle diagram, starting at a number and following one arrow tells us what α does to that number. Following two arrows tells us what α^2 does. So in the circle diagram, if we start at 1 and follow 2 arrows, we must reach 2:

$$\begin{tikzcd}
 & 1 \arrow[dr] & \\
? \arrow[ur] & & ? \arrow[d] \\
? \arrow[u] & & 2 \arrow[dl] \\
 & ? \arrow[ul] &
\end{tikzcd}$$

Since α^2 takes 2 to 3 and 3 to 1, going two arrows from 2 must take us to 3, and following two arrows from 3 must take us to 1:

$$\begin{tikzcd}
& 1 \arrow[dr] & \\
? \arrow[ur] & & ? \arrow[d] \\
3 \arrow[u] & & 2 \arrow[dl] \\
& ? \arrow[ul] &
\end{tikzcd}$$

If we look at the second cycle (456) in α^2, we see that in the circle diagram, if we start at 4 and follow two arrows, we reach 5. Starting at 5 and following two arrows will take us to 6, and following two more arrows should return us to 4. But where does 4 go in the circle diagram? There are three question marks leftover, and it turns out that 4 can go in any of them. Let me choose the question mark in the upper right. Then 5 is two arrows away from 4, and 6 is two arrows away from 5:

$$\begin{tikzcd}
& 1 \arrow[dr] & \\
6 \arrow[ur] & & 4 \arrow[d] \\
3 \arrow[u] & & 2 \arrow[dl] \\
& 5 \arrow[ul] &
\end{tikzcd}$$

This tells us that $\alpha = (142536)$.

We could have also placed 4 in the bottom:

$$\begin{tikzcd}
& 1 \arrow[dr] & \\
5 \arrow[ur] & & 6 \arrow[d] \\
3 \arrow[u] & & 2 \arrow[dl] \\
& 4 \arrow[ul] &
\end{tikzcd}$$

This tells us that $\alpha = (162435)$.

Another choice would be to put 4 in the upper left:

$$
\begin{tikzcd}
 & 1 \arrow[dr] & \\
4 \arrow[ur] & & 5 \arrow[d] \\
3 \arrow[u] & & 2 \arrow[dl] \\
 & 6 \arrow[ul] &
\end{tikzcd}
$$

This tells us that $\alpha = (152634)$.

Overall, there are (at least) three answers to this problem. Essentially, the permutation $(123)(456)$ has (at least) three "square roots."