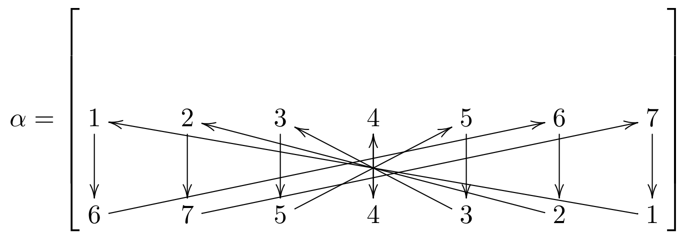
Midterm 2 Solutions

March 17, 2012

Problem 1

Part a)

Let us draw arrows to help us find the cycles:



There are three cycles. First, we have a 4-cycle where 1 goes to 6 goes to 2 goes to 7 goes back to 1. Next, we have a 2-cycle where 3 goes to 5 goes back to 3. 4 is fixed and will be part of a cycle of length one. Thus,

$$\alpha = (1627)(35)(4) = (1627)(35).$$

Part b)

We can find the order of α by looking at the disjoint cycle form and taking the least common multiple of the lengths of the cycles. α is a product of two cycles, one with length 4, and the other with length 2, so

$$|\alpha| = \operatorname{lcm}(4, 2) = 4.$$

Part c)

To determine whether a permutation is odd or even, we need to convert it to a product of 2-cycles and count the number of 2-cycles. A permutation that can be written as an odd (even) number of 2-cycles is odd (even). We have

$$\alpha = (1627)(35) = (17)(12)(16)(35)$$

which is a product of four 2-cycles. Hence α is an even permutation.

Problem 2

Part a)

TRUE: Let a be a nonzero element of $\mathbb{Z}$. Remember that the operation in $\mathbb{Z}$ is addition, so when computing the order of a, we need to look for a positive nonzero integer n such that

$$n \cdot a = 0.$$

If there is no such integer n, then a is said to have infinite order.

Since there are no positive nonzero integers that we can plug into n to get the left side equal to zero (since a is nonzero), a has infinite order.

On the other hand, 0 itself has order 1 since $1 \cdot 0 = 0$.

Part b)

FALSE: By Theorem 4.2, page 75,

$$\left|a^5\right| = \frac{|a|}{\gcd(|a|, 5)} = \frac{15}{\gcd(15, 5)} = \frac{15}{5} = 3 \neq 5.$$

Part c)

FALSE: Suppose for contradiction that ϕ is an automorphism. Then it must preserve order:

$$|\phi(3)| = |3|.$$

First, we must note that $\mathbb{Z}_{100}$ is a cyclic group (operation is addition mod 100) generated by 1. That is, $|1| = 100$ in $\mathbb{Z}_{100}$. Thus, since $3 = 3 \cdot 1$,

$$|3| = \frac{|1|}{\gcd(|1|, 3)} = \frac{100}{\gcd(100, 3)} = \frac{100}{1} = 100.$$

However,

$$|\phi(3)| = |40| = \frac{|1|}{\gcd(|1|, 40)} = \frac{100}{\gcd(100, 40)} = \frac{100}{20} = 5 \neq 100 = |3|,$$

a contradiction. Thus, ϕ is not an automorphism.

Part d)

TRUE: Let us try a proof by contradiction and suppose $|G|$ is even. Then 2 divides $|G|$. Since G is a finite cyclic group, we can use Theorem 4.4, page 79, to see that G has $\phi(2) = 1$ element of order 2, contradicting the fact that G has no elements of order 2. Hence $|G|$ is odd.

Problem 3

Since G is a cyclic group and 10 divides $|G|$, we can use Theorem 4.4, page 79 to determine the number of elements of order 10 in G. It is

$$\phi(10) = 4.$$

Problem 4

Here are multiple ways to solve this problem. All of them proceed using a proof by contradiction where we assume there is an isomorphism and show that there is a contradiction using Theorems 6.2 and 6.3 on pages 128 and 129.

Using Theorem 6.3

Suppose that $\phi : U(8) \to \mathbb{Z}_4$ is an isomorphism. By Theorem 6.3, since $\mathbb{Z}_4$ is cyclic, then so is $U(8)$, which is false. Hence, there is no isomorphism from $U(8)$ to $\mathbb{Z}_4$.

Using Theorem 6.2, Property 7

Suppose that $\phi : U(8) \to \mathbb{Z}_4$ is an isomorphism. $U(8)$ and $\mathbb{Z}_4$ are both finite, so by Theorem 6.2, property 7, $U(8)$ and $\mathbb{Z}_4$ have the same number of elements of order 2. But $U(8)$ has three elements of order 2 (3, 5, and 7), while $\mathbb{Z}_4$ only has one element of order 2 (the integer 2), a contradiction.

Using Theorem 6.2, Properties 1 and 5

Suppose that $\phi : U(8) \to \mathbb{Z}_4$ is an isomorphism. Let us write down what ϕ does to each number in $U(8)$. First, $\phi(1) = 0$, because 1 is the identity element of $U(8)$ and 0 is the identity element of $\mathbb{Z}_4$. Now, ϕ is one-to-one and onto, so we know there is only one element $a \in U(8)$ for which $\phi(a) = 2$. We note that $a \neq 1$ (since we already have $\phi(1) = 0$), and $|a| = 2$ since every element other than 1 has order 2 in $U(8)$. Let b be another element of $U(8)$ not equal to a or 1. Then b has order 2 as well, so $\phi(b)$ must have order 2 in $\mathbb{Z}_4$. Since the only element of order 2 in $\mathbb{Z}_4$ is 2, we must have $\phi(b) = 2$, contradicting the assumption that ϕ is a one-to-one function.

Problem 5

We can prove that A_6, the subgroup of S_6 consisting of even permutations, has no element of order 6 by:

1. Finding all elements in S_6 with order 6, using disjoint cycle structures to keep track of elements, and then

2. Showing that these permutations with order 6 are odd permutations, so they are not in A_6.

In S_6, the only elements whose cycle lengths have lcm 6 are of the form

$$(\underline{6})$$

(a single cycle of length 6), and

$$(\underline{3})(\underline{2})(\underline{1})$$

(a product of three disjoint cycles: a 3-cycle, a 2-cycle, and a 1-cycle). These are in S_6 because the sum of the lengths of the disjoint cycles is equal to six.

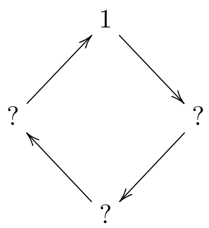
Now we need to check if these are even or odd. In general, an n-cycle can be written as a product of n-1 2-cycles. Thus,

- A single cycle of length 6 is a product of five 2-cycles, and hence is odd.
- A product of a 3-cycle, a 2-cycle, and a 1-cycle is equal to a product of three 2-cycles. The 3-cycle gives us two 2-cycles, the 2-cycle gives us one 2-cycle (itself), and the 1-cycle gives us zero 2-cycles. Hence it is odd as well.

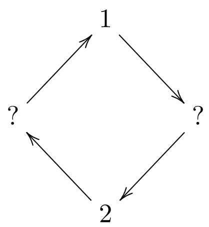
Hence any element of order 6 in S_6 can be written as an odd number of 2-cycles, so they are odd and cannot be in A_6.

Problem 6

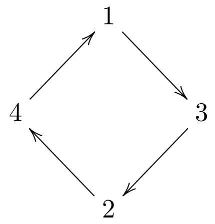
Let α be a 4-cycle for which $\alpha^2 = (12)(34)$. To find α, it helps to start with a circle diagram. We know 1 has to appear in the cycle, since a 4-cycle on $\{1, 2, 3, 4\}$ must use all four numbers:



α^2 takes 1 to 2. On the circle diagram, starting at a number and following one arrow tells us what α does to that number. Following two arrows tells us what α^2 does. So in the circle diagram, if we start at 1 and follow two arrows, we must reach 2:

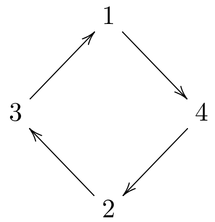


If we look at the second cycle (34) in α^2, we see that in the circle diagram, if we start at 3 and follow two arrows, we reach 4. Starting at 4 and following two arrows will take us back to 3. But where does 3 go in the circle diagram? There are two question marks leftover, and it turns out that 3 can go in any of them. Let me choose the question mark in the right. Then 4 is two arrows away from 3:



This tells us that $\alpha = (1324)$.

We could have also placed 3 in the left:



This tells us that $\alpha = (1423)$ is another possible answer.

Problem 7

$\phi_x = \phi_y$ implies $x = y$

Suppose that $\phi_x = \phi_y$. Then for any $g \in G$,

$$\phi_x(g) = \phi_y(g).$$

By definition of the functions,

$$xgx^{-1} = ygy^{-1}.$$

We need to show that $x = y$. Our only clue in this problem is that the center of G is $\{e\}$. That is, the only element in G which commutes with every other element in G is the identity. Let us try to show $x = y$ by somehow proving that $y^{-1}x = e$.

Since

$$xgx^{-1} = ygy^{-1},$$

let us multiply on the left by y^{-1} and on the right by x:

$$y^{-1}xgx^{-1}x = y^{-1}ygy^{-1}x$$

$$(y^{-1}x)g = g(y^{-1}x).$$

This is true for any element $g \in G$. Hence $y^{-1}x$ is in the center of G:

$$y^{-1}x \in Z(G) = \{e\}.$$

Therefore, $y^{-1}x = e$, so $x = y$.

$x = y$ implies $\phi_x = \phi_y$

If $x = y$, then $x^{-1} = y^{-1}$, so for any $g \in G$,

$$\phi_x(g) = xgx^{-1} = ygy^{-1} = \phi_y(g).$$

Therefore, $\phi_x = \phi_y$.