Review in class on Friday

Previously in Math 103a:

Order of a group $G = |G|$ = number of elements in G

Order of an element $g \in G = |g|$
= smallest positive integer n for which $g^n = e$

$$\boxed{\langle g \rangle = \{ g^m : m \in \mathbb{Z} \}}$$

Note that $\langle g \rangle$ is a group & $\langle g \rangle \subset G$

Subgroups

Def. A subset H of a group G that is itself a group (with same mult as G) is called a subgroup of G

Notation: $\boxed{H \leq G}$ means H is a subgroup of G. If $H \neq G$, write $\boxed{H < G}$

If $H \neq \{e\}$, H is called a nontrivial subgroup of G.

Ex.1 $\boxed{\{2n : n \in \mathbb{Z}\}}$ is a subgroup of $\mathbb{Z}$ under operation of addition.

Ex 2 $\boxed{\{0, 1, 2, \ldots, n-1\}}$ with addition mod n is a group, is a subset of $\mathbb{Z}$, but is not a subgroup, because operation is not same.

e.g. $\boxed{\begin{array}{ll} \text{in } \mathbb{Z} & (n-1)+1 = n \\ \text{in } \mathbb{Z}_n & (n-1)+1 = 0 \end{array}}$

Ex 3 G any group, $g \in G$, then $\boxed{\langle g \rangle}$ is a subgroup.

Ex 4 $\mathbb{Z}^+ \cup \{0\} = \{0, 1, 2 \cdots\}$ is not a subgroup of $\mathbb{Z}$ (addition)

$$n, m \in \mathbb{Z}^+ \cup \{0\} \Rightarrow n+m \in \mathbb{Z}^+ \cup \{0\}$$

but no positive integer has an inverse.

Ex 5. $\mathbb{Q}^+$ (= all positive rational numbers) is a proper subgroup of $\mathbb{Q}^*$.

Q: How to tell when a subset is a subgroup?

Thm (One step subgroup test). Let G group, $H \subset G$, $H \neq \emptyset$. Then H is a subgroup of $G \Longleftrightarrow$ if $a, b \in H$, then $ab^{-1} \in H$.

Pf: H subgroup, $a, b \in H \Rightarrow ab^{-1} \in H$.

For other direction, suppose $a, b \in H \Rightarrow ab^{-1} \in H$. Check assoc, identity & inverses.

Assoc. holds since operation is same as in G, (and assoc. holds in G).

Identity. $H \neq \emptyset \Rightarrow \exists\, x \in H$. Take $a=b=x$. Then $ab^{-1} = xx^{-1} = e \in H$ by hypothesis.

Inverses: Let $x \in H$. Need to show $x^{-1} \in H$. Take $\boxed{\begin{matrix} a=e \\ b=x \end{matrix}}$ (which is in H by above) and. Then $ab^{-1} = ex^{-1} = \boxed{x^{-1} \in H}$.

Closed under mult: Let $x, y \in H$. Need to show $xy \in H$. Take $a = x$, $b = y^{-1}$ (in H by above). Then $ab^{-1} = x(y^{-1})^{-1} = xy \in H$. //

Ex 1. G abelian, $H = \{x \in G : x^2 = e\}$

$\boxed{\begin{matrix} H \neq \emptyset \\ e \in H \end{matrix}}$

Check: $x, y \in H \Rightarrow x^2 = e,\ y^2 = e$
$\Rightarrow x^{-1} = x,\ y^{-1} = y$

$\therefore (xy^{-1})^2 = xy^{-1}xy^{-1} = xyxy$
$= x^2y^2 = ee = e$.

abelian

$\therefore \boxed{xy^{-1} \in H}$, so H is a subgroup by one step test.

Ex 2. G any group, $H = \{x \in G : xa = ax\ \ \forall a \in G\}$

H is called the center of G.

H is a subgroup. $H \neq \emptyset$ since $e \in H$.

Use one step test:

Note that if $y \in H$,

$ya = ay \Longrightarrow a = y^{-1}ay \Longrightarrow ay^{-1} = y^{-1}a$

$\therefore\ y^{-1} \in H$.

Suppose $x, y \in H$. Then $\boxed{(xy^{-1})a = x(y^{-1}a) = x(ay^{-1}) = (xa)y^{-1} = (ax)y^{-1} = a(xy^{-1})}$

$\therefore\ xy^{-1} \in H$, so H is a subgroup.

Ex 3. Find the center of D_n, written $Z(D_n)$

R_0 = identity is in center.

Other rotations? Suppose R rotation is in center

Then $RF = FR$ for all flips F in D_n.

RF is again a flip, so $\boxed{(RF)^{-1} = RF}$.

Also, $(RF)^{-1} = F^{-1}R^{-1}$ (products)

$= FR^{-1}$

$\therefore\ FR = RF = FR^{-1} \Longrightarrow R = R^{-1}$

$\Longrightarrow R = R_{180^\circ}$

$\therefore$ only possible rotations in center are R_0, R_{180}

Also, if F is any flip, $FR_\alpha \neq R_\alpha F$ unless

$\alpha = 0$ or 180. $\therefore$ no flip is in center.

If n is even R_{180} is a symmetry, so $Z(D) = \{R_0, R_{180}\}$

If n is odd $R_{180} \notin D_n$, so $Z(D) = \{R_0\}$

Ex 4 $G = D_n$, $H =$ all rotations in D_n

H is a subgroup, since

$$H = \left\{ R_{\frac{360}{n}\cdot k},\ k \in \mathbb{Z} \right\}$$

$$R_\alpha (R_\beta)^{-1} = R_\alpha R_{360^\circ - \beta} = R_{360+(\alpha-\beta)}$$

$$\alpha = \frac{360}{n} k, \quad \beta = \frac{360}{n} \ell$$

$$\alpha - \beta = \frac{360}{n}(k - \ell)$$

Ex 5 $G = D_n$, $H = \{$ Reflections in $D_n \}$

H is not a subgroup,

although $x \in H \Rightarrow x^{-1} \in H$, since $x = x^{-1}$

But $x\underbrace{y^{-1}}_{xy}$ is a rotation if $x, y \in H$.

$\therefore xy \notin H$. (H also lacks identity, but is still not a group even if identity is added.)

Two Step subgroup Test.

Thm. G group, $H \neq \emptyset$. If H is closed under group mult, i.e. $\boxed{a, b \in H \Rightarrow ab \in H}$, and $\boxed{a^{-1} \in H \text{ for all } a \in H}$, then H is a subgp.

Pf: Use 1 step. Let $a, b \in H$. By hypothesis, $b^{-1} \in H$. By closed, $ab^{-1} \in H$. $\therefore H$ is a subgroup by 1-step.