Previously in Math 103a:

Examples of isomorphisms of groups. $\varphi : G \longrightarrow \bar{G}$

Today: We'll show that every group is isomorphic to group of permutations.

Def A permutation of a set A is a bijection $\alpha : A \longrightarrow A$ (1-1 & onto)

A group of permutations or a permutation group is a subgroup of the group of all permutations of some set.

group operation for permutations is composition: $\alpha\beta : A \longrightarrow A$ given by

$$\boxed{A \xrightarrow{\beta} A \xrightarrow{\alpha} A.}$$

First, finish examples of isomorphisms. (last page of previous lecture)

Identifying group elements with permutations

Ex. $G = \mathbb{Z}_3 = \{0, 1, 2\}$ addition mod 3

Think of every $a \in G$ as a permutation T_a of $\{0, 1, 2\}$

$$\boxed{\begin{aligned} T_0(0) &= 0 \\ T_0(1) &= 1 \\ T_0(2) &= 2 \end{aligned}}$$

T_0 is identity permutation

$$\boxed{\begin{aligned} T_1(0) &= 0+1 = 1 \\ T_1(1) &= 1+1 = 2 \\ T_1(2) &= (2+1) \bmod 3 = 0 \end{aligned}}$$

Notation:

$$\begin{matrix} 0 & 1 & 2 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 0 \end{matrix}$$

$$\boxed{T_1 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}}$$

$$\boxed{\begin{aligned} T_2(0) &= 0+2 = 2 \\ T_2(1) &= (1+2) \bmod 3 = 0 \\ T_2(2) &= (2+2) \bmod 3 = 1 \end{aligned}}$$

$$\boxed{T_2 = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}}$$

$\bar{G} = \{T_0, T_1, T_2\} \subset$ all permutations of $\{0, 1, 2\}$

How many permutations of $\{0,1,2\}$?

3 choices for $\alpha(0)$
2 choices for $\alpha(1)$
1 choice for $\alpha(2)$

$3 \times 2 \times 1 = 6$ permutations of $\{0, 1, 2\}$

$\mathbb{Z}_3 \approx$ subgroup of order 3.

More notation:

If $A = \{1, 2, \ldots, n\}$, then any permutation $\alpha : A \longrightarrow A$ may be expressed as

$$\boxed{\begin{bmatrix} 1 & 2 & 3 & \cdots\cdots & n \\ \alpha(1) & \alpha(2) & \alpha(3) & \cdots & \alpha(n) \end{bmatrix}}$$

Ex $A = \{1, 2, 3, 4\}$, $\alpha : A \longrightarrow A$

written $\begin{bmatrix} 1 & 2 & 3 & 4 \\ \alpha(1) & \alpha(2) & \alpha(3) & \alpha(4) \end{bmatrix}$

e.g if $\alpha(1) = 2$, $\alpha(2) = 3$, $\alpha(3) = 1$, $\alpha(4) = 4$

Then $\boxed{\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{bmatrix}}$

The set of all permutations of a set A forms a group under composition

Ex: $A = \{1, 2, 3\}$, $\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$, $\beta = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$

Then $\boxed{\alpha\beta = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}}$

$1 \longmapsto 1 \longmapsto 2$
$2 \longmapsto 3 \longmapsto 3$
$3 \longmapsto 2 \longmapsto 1$

$\alpha\beta(1) = 2$, $\alpha\beta(2) = 3$, $\alpha\beta(3) = 1$.

Notation: Put $\boxed{S_n}$ for the group of permutations of $\{1, 2, \ldots, n\}$ i.e. any set of n objects.

Thm Any group G is isomorphic to a group of permutations. More precisely, if $|G| = n$ (or ∞) then $\boxed{G \approx \bar{G} \leq S_n}$ (or S_∞).

Do it for $G = U(12) = \{1, 5, 7, 11\}$

put $\varphi(j) = T_j$ for $j = 1, 5, 7, 11$

where $\boxed{T_j(k) = (jk) \bmod 12}$, for $k \in \{1, 5, 7, 11\}$

Put $\bar{G} = \{T_1, T_5, T_7, T_{11}\} \subset S_4$

We can write $\alpha \in S_4$ in form $\begin{bmatrix} 1 & 5 & 7 & 11 \\ \alpha(1) & \alpha(5) & \alpha(7) & \alpha(11) \end{bmatrix}$

Then $\boxed{T_1 = \begin{bmatrix} 1 & 5 & 7 & 11 \\ 1 & 5 & 7 & 11 \end{bmatrix}}$ since $(1\,k) \bmod 12 = k$

$T_5(1) = 5 \cdot 1 = 5$, $T_5(5) = (25) \bmod 12 = 1$

$T_5(7) = 35 \bmod 12 = 11$, $T_5(11) = 55 \bmod 12 = 7$

$\therefore \boxed{T_5 = \begin{bmatrix} 1 & 5 & 7 & 11 \\ 5 & 1 & 11 & 7 \end{bmatrix}}$

$T_7(1) = 7$, $T_7(5) = 11$, $T_7(7) = 1$, $T_7(11) = 5$

$\therefore \boxed{T_7 = \begin{bmatrix} 1 & 5 & 7 & 11 \\ 7 & 11 & 1 & 5 \end{bmatrix}}$

$T_{11}(1) = 11$, $T_{11}(5) = 7$, $T_{11}(7) = 5$, $T_{11}(11) = 1$

$\boxed{T_{11} = \begin{bmatrix} 1 & 5 & 7 & 11 \\ 11 & 7 & 5 & 1 \end{bmatrix}}$

check that $\varphi: j \longmapsto T_j$ is 1-1

$$T_j = T_{j'} \Rightarrow T_j(1) = T_{j'}(1) \Rightarrow j = j'$$

$\therefore$ 1-1. Onto $\bar{G}$ is clear: $j \to T_j$

φ Isomorphism? First: is $\{T_1, T_5, T_7, T_{11}\} = \bar{G}$ a group under composition?

$$T_j T_\ell(h) = T_j(T_\ell(h)) = T_j(\ell h \bmod 12)$$

$$= (j\ell h) \bmod 12 = (((j\ell) \bmod 12)h) \bmod 12$$

$$= (mh) \bmod 12, \text{ where } j\ell \bmod 12 = m$$

$=$

$$(mh) \bmod 12 = T_m(h)$$

True for $h = 1, 5, 7, 11$

This shows $\bar{G}$ is a group

$$\therefore \varphi(j)\varphi(\ell) = \varphi(m),$$

where $m = j\ell$ in $U(12)$

This shows φ preserves group operations

$\therefore \varphi$ is an isomorphism from $G \to \bar{G}$

General case is essentially the same

For any group G, let $\bar{G} = \{T_g : g \in G\}$

where $\boxed{T_g(x) = gx}$ for $x \in G$

(left regular representation)

Then T_g is a group with

$$\boxed{T_g T_h(x)} = T_g(hx) = g(hx) = (gh)x$$
$$= \boxed{T_{gh}(x)}$$

$\therefore \quad \varphi : G \longrightarrow \overline{G}$

$\qquad g \longmapsto T_g$

is an isomorphism.

Back to permutations (which are really interesting!)

Note that S_n is isomorphic to a subgroup of S_{n+1}: if $\alpha \in S_n$, define $\varphi(\alpha) \in S_{n+1}$

$$\varphi(\alpha)(k) = \alpha(k) \quad k = 1, 2, \ldots, n$$
$$\& \quad \varphi(\alpha)(n+1) = n+1$$

$$S_n \approx \{ \beta \in S_{n+1} : \beta(n+1) = n+1 \}$$

<u>Thm.</u> S_n is nonabelian for $n \geq 3$.

Pf: By above, enough to show it for $n = 3$ (a subgroup of an abelian group must be abelian)

S_3 nonabelian: $\quad \alpha = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$

Then $\alpha\beta = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \qquad \beta\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ //