MO

Previously in Math 103a

Permutation: $\alpha : A \longrightarrow A$ bijection

$\{\alpha : A \to A\}$ is a group under composition of operators

Notation: $A = \{1, 2, \ldots, n\}$

Then write $\boxed{\alpha = \begin{bmatrix} 1 & 2 & \cdots & n \\ \alpha(1) & \alpha(2) & & \alpha(n) \end{bmatrix}}$

S_n = group of all permutations of $\{1, 2, \ldots, n\}$ (degree n)

We proved:

Thm S_n is nonabelian for $n \geq 3$.

Also useful:

Thm $\boxed{|S_n| = n!} = n(n-1)(n-2)\cdots 1$

Sketch of Pf: To determine $\alpha \in S_n$, there are n choices for $\alpha(1)$. Fix one choice. Then $\exists$ $n-1$ choices for $\alpha(2)$. Fixing $\alpha(1)$ & $\alpha(2)$ leaves $n-2$ choices for $\alpha(3)$. Continuing, $\alpha \in S_n$ can be defined in $\boxed{n!}$ different ways. //

Today: Cycles!

Cycles

Ex 1. $\boxed{\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}}$

Think of this as $1 \longrightarrow 3 \longrightarrow 1$, 2 fixed

Cycle notation is $\boxed{(1\,3)}$ or $\boxed{(3\,1)}$

Ex 2. $\boxed{\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}}$

$$1 \longrightarrow 3 \longrightarrow 2$$

In cycle notation $\boxed{\alpha = (1\,3\,2)} = (2\,1\,3) = (3\,2\,1)$

$$(1\,3\,2) = (1, \alpha(1), \alpha^2(1)).$$

Def. a cycle is a permutation of the form

$$k \longrightarrow \alpha(k) \longrightarrow \alpha^2(k) \longrightarrow \cdots \longrightarrow \alpha^p(k) = k$$

Not necessary to know this definition

where α is a permutation & the $\alpha^j(k)$ are all distinct for $j < p$,
i.e. follow k until it returns to itself.

All permutations in S_3 are cycles, but not true in S_4

Ex $n = 4$ $\boxed{\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}}$

$1 \longrightarrow 2$ $\quad$ $3 \longrightarrow 4$ $\quad$ $\boxed{\alpha = (1\,2)(3\,4)}$

product of cycles

Products of cycles

Multiplication of cycles = composition of operators

Ex 1 $n=5$ $\alpha=(135)(234)=?$

first apply (234):

$1\to1$
$2\to3$
$3\to4$
$4\to2$
$5\to5$

Now apply (135):

$1\to1\to3$
$2\to3\to5$
$3\to4\to4$
$4\to2\to2$
$5\to5\to1$

$$(135)(234)=\begin{bmatrix}1&2&3&4&5\\3&5&4&2&1\end{bmatrix}$$

Start with α & write as product of cycles.

$$1\to3\to4\to2\to5$$

$\therefore$ $\alpha=(13425)$ single cycle

Ex 2. $n=6$ $(41)(36)(34)(13)(25)=\alpha$

start from right

$1\to3\to4\to1$ so $\alpha(1)=1$
$2\to5$ $\alpha(2)=5$
$3\to1\to4$ $\alpha(3)=4$
$4\to3\to6$ $\alpha(4)=6$
$5\to2$ $\alpha(5)=2$
$6\to3$ $\alpha(6)=3$

$$\alpha=\begin{bmatrix}1&2&3&4&5&6\\1&5&4&6&2&3\end{bmatrix}$$

As disjoint cycles:

$(1)(25)(346)=\alpha$

omit this

Notation: write ε for identity permutation

Thm Any permutation of a finite set A can be written as a cycle or a product of disjoint cycles.

pf + method: Start with any element of $A = \{1, 2, \ldots, n\}$, say 1. Apply α successively to 1: (*) $\{1, \alpha(1), \alpha^2(1), \ldots, \alpha^{k-1}(1)\}$, where k is the smallest number for which $\alpha^k(1) = 1$.

(Such a k exists, since $|\alpha| < \infty$.)

Claim $\alpha^j(1) \neq \alpha^\ell(1)$ for $0 \leq \ell < j < k$

i.e. all the numbers in (*) are distinct.

If $\alpha^j(1) = \alpha^\ell(1)$, then $\alpha^{j-\ell}(1) = 1$, contradicting the choice of k, since $j < k$. Hence

$$\boxed{\sigma = (1, \alpha(1), \ldots, \alpha^{k-1}(1))} \text{ is a cycle.}$$

If every $J \in \{1, 2, \ldots, n\}$ is of the form $\alpha^p(1)$ for some p, $1 \leq p < n$, then done.

Otherwise, take smallest $J \notin \{1, \alpha(1), \ldots, \alpha^{k-1}(1)\}$. Let k_2 be smallest for which $\alpha^{k_2}(J) = J$.

Then $(J, \alpha(J), \alpha^2(J), \ldots, \alpha^{k_2-1}(J))$ is a cycle & is disjoint from σ. Continue … //

until every number is in a cycle

Ex Write $(1235)(24567)$ as a product of disjoint cycles

Ans. $1 \to 2 \to 4 \qquad 3 \to 5 \to 6 \to 7$

so $\boxed{(1235)(24567) = (124)(3567).}$

Check by doing it a different way:

put $\alpha = (1235)$, $\beta = (24567)$

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 5 & 4 & 1 & 6 & 7 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 3 & 5 & 6 & 7 & 2 \end{bmatrix}$$

$$\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 1 & 6 & 7 & 3 \end{bmatrix}$$

$$= (1\ 2\ 4)(3\ 5\ 6\ 7) \checkmark$$

How to find $|\alpha|$, $\alpha \in S_n$.

1. The order of a cycle

$$\sigma = (J, \alpha(J), \alpha^2(J), \ldots, \alpha^{k-1}(J))$$

$\boxed{|\sigma| = k}$ = length of cycle

Reason: $\alpha^k(J) = J$, by definition

$$\alpha^k(\alpha^j(J)) = \alpha^j(\alpha^k(J)) = \alpha^j(J)$$

$\therefore\ \alpha^k(x) = x \quad \forall\ x$ in cycle.

If $x \notin$ cycle, then $\alpha(x) = x \Rightarrow \alpha^k(x) = x.$

MS

2. Disjoint cycles commute.

Reason: disjoint cycles permute disjoint sets, so it doesn't matter which you apply first.

3. If σ & τ are disjoint cycles in S_n, then

$$\boxed{|\sigma\tau| = \mathrm{lcm}(|\sigma|, |\tau|)}$$

Reason: For any m, $(\sigma\tau)^m = \sigma^m\tau^m$ by 2. Since σ^m & τ^m permute disjoint subsets of $\{1, 2, \ldots, n\}$,

$\sigma^m\tau^m = \varepsilon$ (identity) $\Longleftrightarrow$ $\sigma^m = \varepsilon$ & $\tau^m = \varepsilon$.

$$\boxed{\begin{array}{c} \sigma^m = \varepsilon \Longleftrightarrow |\sigma| \,\big|\, m \\ \tau^m = \varepsilon \Longleftrightarrow |\tau| \,\big|\, m \\ \therefore (\sigma\tau)^m = \varepsilon \Longleftrightarrow |\sigma| \,\big|\, m \ \& \ |\tau| \,\big|\, m \end{array}}$$

$\therefore$ $|\sigma\tau|$ = smallest $m > 0$ for which $|\sigma| \,\big|\, m$ & $|\tau| \,\big|\, m$ $\therefore$ $|\sigma\tau| = \mathrm{lcm}(|\sigma|, |\tau|)$.

4. To find the order of a permutation α, write $\alpha = \sigma_1\sigma_2 \ldots \sigma_p$ product of disjoint cycles.

Then $|\alpha| = \mathrm{lcm}(|\sigma_1|, |\sigma_2| \ldots |\sigma_p|)$

Ex. Find the order of

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{bmatrix}$$

Ans: $\alpha = (1\,2\,5)(3\,4)$ $\qquad |(1\,2\,5)| = 3$

$|(3\,4)| = 2$

$\therefore |\alpha| = 6$