Previously in Math 103a

Cycles: $\boxed{(a_1, a_2, \ldots a_k)} \in S_n$, $n \geq k$, $a_j \neq a_k$ if $j \neq k$

represents the permutation

$$\boxed{\begin{array}{l} a_1 \to a_2 \\ a_2 \to a_3 \\ a_3 \to a_4 \\ \vdots \\ a_{k-1} \to a_k \\ a_k \to a_1 \end{array}}$$

& all other elements are left fixed.

We proved the following:

(1) Any $\alpha \in S_n$ is a product of disjoint cycles (unique representation)

(2) If σ_1, σ_2 are disjoint cycles, then

$$\boxed{\sigma_1 \sigma_2 = \sigma_2 \sigma_1}$$

(3) If σ is a cycle of length k, then

$$\boxed{|\sigma| = k}$$

(4) If $\alpha = \sigma_1 \sigma_2 \ldots \sigma_\ell$ where the σ_i are disjoint cycles, then

$$\boxed{|\alpha| = \operatorname{lcm}(|\sigma_1|, |\sigma_2|, \ldots, |\sigma_\ell|)}$$

We'll prove:

Thm. Any $\alpha \in S_n$ is a product of 2-cycles (not unique representation)

This is easy to do. Terminology: transposition = 2-cycle

Ex: write $\boxed{\sigma = (21634)} \in S_6$ as a product of 2-cycles.

Ans. $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 4 & 2 & 5 & 3 \end{bmatrix}$

start with $1 \to 6$

$$\boxed{\sigma = (12)(14)(13)(16)}$$

We could do it another way: start with $2 \to 1$

$$\boxed{\sigma = (24)(23)(26)(21)}$$

Observation: In both representations, σ is a product of 4 transpositions

this is not the only possibility!

Could write $\boxed{\sigma = (34)(34)(24)(23)(26)(21)}$

since $(34)(34) = \varepsilon$.

Amazing fact: in any representation a cycle of odd order is the product of an even number of transposition

More precisely:

Thm. Any permutation $\alpha \in S_n$ is a product of k transpositions with $k < n$. If α is also a product of l transpositions, $l > 0$, then $k + l$ is even, i.e. if k is even, then l is even, if k is odd, then l is odd. In other words, k & l have the same parity.

The proof reduces to this lemma

Lemma. If $\varepsilon = \beta_1 \beta_2 \cdots \beta_r$, where the β_i are 2 cycles, then r is even.

Pf of Thm (assuming Lemma).

Suppose $\alpha = \beta_1 \beta_2 \cdots \beta_k = \gamma_1 \gamma_2 \cdots \gamma_l$

where the β_i & γ_j are 2-cycles

Then $\beta_1 \beta_2 \cdots \beta_k \gamma_l \gamma_{l-1} \cdots \gamma_1 = \varepsilon$.

since $\gamma_l \gamma_{l-1} \cdots \gamma_1 = (\gamma_1 \gamma_2 \cdots \gamma_l)^{-1}$

since $\gamma_i^{-1} = \gamma_i$ for 2-cycles.

By Lemma, $k + l$ is even. $//_{\text{Thm}}$

Need to prove Lemma ...

Pf of Lemma. Not so easy! Use induction on r. First $r \neq 1$, since no 2-cycle is the identity. If $r=2$, then done. So suppose $r > 2$.

Look at last transposition β_r in representation $\varepsilon = \beta_1 \beta_2 \cdots (\beta_{r-1} \beta_r)$

Idea: Suppose $\beta_r = (a, b)$. Try to push all occurrences of a to the far left. The only occurrence of a cannot be in the cycle at the far left since this could not be the identity. After pushing, the # of 2-cycles in the representation of ε will be reduced by a positive & even number of 2-cycles.

Ex of method: $\varepsilon = (23)(13)(14)(13)(34)(23)$

Take $a=2$ rewrite $(34)(23)$ $\quad 2 \to 3 \to 4, \quad 3 \to 2$

$= (24)(34)$ replace

Now $\varepsilon = (23)(13)(14)(13)(24)(34)$

$(13)(24) = (24)(13)$ disjoint

replace

$\varepsilon = (23)(13)(14)(24)(13)(34)$

$$(14)(24) = (21)(41) \quad \text{replace}$$

$$\varepsilon = (23)(13)(21)(41)(13)(34)$$

$$(13)(21) = (23)(13) \quad \text{replace}$$

$$\varepsilon = (23)(23)(13)(41)(13)(34)$$

$$\varepsilon = \underline{(13)(41)(13)(34)} \quad \text{length } r-2$$

No 2 appears & there are 2 fewer 2-cycles.

Now result follows by induction on r.

See text for more precise details. //

Def A permutation is called even if it can be written as the product of an even number of 2-cycles. It is called odd if it can be written as the product of an odd number of 2-cycles.

Easy to check:

Thm. The set of even permutations forms a subgroup A_n of S_n.

Notation: A_n is called the alternating subgroup of S_n.

Observation: (1) The product of 2 even permutations is even

(2) The product of 2 odd permutations is even

(3) The product of an even permutation and an odd permutation is odd.

Another interpretation of odd & even via permutation matrices (not in book)

Def An $n \times n$ matrix is a permutation matrix if it contains exactly one 1 in each row & each column with all other entries $= 0$.

Ex: $n = 4$ $\quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a permutation matrix

It corresponds to the permutation given by the cycle $(1\,3\,2) \in S_4$,

since $A\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}$ $\quad$ $1 \to 3$, $2 \to 1$, $3 \to 2$, $4 \to 4$

$\det A = 1$

Every $\alpha \in S_n$ corresponds to a unique permutation matrix A_α $n \times n$ define by $A_\alpha = (a_{jk})$ where

$$\boxed{a_{jk} = \begin{cases} 1 & \text{if } \alpha(j) = k \\ 0 & \text{otherwise} \end{cases}}$$

Composition of permutations corresponds to matrix multiplication of permutation matrices!

$$\boxed{\begin{array}{ll} \text{If} & \alpha \longmapsto A \\ & \beta \longmapsto B \\ \text{Then} & \alpha\beta \longmapsto AB \end{array}} \qquad \begin{array}{l} \det A = \pm 1 \\ \det B = \pm 1 \end{array} \quad \text{why?}$$

Thm: The mapping $\alpha \longmapsto A_\alpha$ is an isomorphism of $\boxed{S_n \text{ onto a subgroup of } GL(n, \mathbb{Z})}$

and $\det A_\alpha = 1 \iff \alpha$ is an even permutation

$\det A_\alpha = -1 \iff \alpha$ is an odd permutation

Idea of pf: The sign of the determinant of a matrix changes if 2 rows are interchanged. Interchanging rows corresponds to composition of α with a 2-cycle. The identity matrix may be obtained from a permutation matrix by interchanging rows.