Previously in Math 103a

$G, \bar{G}$ groups, $\varphi : G \to \bar{G}$ is a **homomorphism** if $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$
$\forall\, g_1, g_2 \in G$.

Various facts about homomorphisms:

(d) $\varphi(G)$ is a subgroup of $\bar{G}$

Reason: $\varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2) \in \varphi(G)$
$\varphi(g_1)^{-1} = \varphi(g_1^{-1})$

so contains products & inverses
(2 step subgroup test)

(e) $\boxed{\varphi(a) = \varphi(b)} \Leftrightarrow ab^{-1} \in \operatorname{Ker}\varphi \Leftrightarrow a^{-1}b \in \operatorname{Ker}\varphi$
$\Leftrightarrow \boxed{a \operatorname{Ker}\varphi = b \operatorname{Ker}\varphi}$ (same coset)

Reason:

$ab^{-1} \in \operatorname{Ker}\varphi \Leftrightarrow \varphi(ab^{-1}) = e \Leftrightarrow \varphi(a)\varphi(b^{-1}) \in \operatorname{Ker}\varphi$
$\Leftrightarrow \varphi(a)\varphi(b^{-1}) = \bar{e} \Leftrightarrow \varphi(a) = \varphi(b)$

$a \operatorname{Ker}\varphi = b \operatorname{Ker}\varphi \Leftrightarrow \exists\, k_1, k_2 \in \operatorname{Ker}\varphi$ with $ak_1 = bk_2$
$\Leftrightarrow a^{-1}b = k_1 k_2^{-1} \in \operatorname{Ker}\varphi$
$\Leftrightarrow b^{-1}a = k_2 k_1^{-1} \in \operatorname{Ker}\varphi$

Similarly, $\varphi(a) = \varphi(b) \Leftrightarrow b^{-1}a \in \ker\varphi$, which gives $a^{-1}b \in \operatorname{Ker}\varphi$.

Ex. $\boxed{\varphi : \mathbb{Z}_{15} \longrightarrow \mathbb{Z}_{15}}$ given by $\boxed{\varphi(x) = 5x}$

$$\operatorname{Ker}\varphi = \{x : 5x \pmod{15} = 0\} = 3\,\mathbb{Z}_{15}$$
$$= \{0, 3, 6, 9, 12\}.$$

Facts about subgroups:

$H \subset G$ subgroup, $K \subset \bar{G}$ subgp.

(A) $\varphi(H)$ is a subgroup of $\bar{G}$ Check it!

$\varphi^{-1}(K) = \{g \in G : \varphi(g) \in K\}$ is a subgp of G

Reason: $g_1, g_2 \in \varphi^{-1}(K) \iff \varphi(g_1), \varphi(g_2) \in K$

$\Longrightarrow \varphi(g_1)\varphi(g_2) \in K$ (since K is a subgroup)

$\Longrightarrow \varphi(g_1 g_2) \in K$

$\therefore\ g_1 g_2 \in \varphi^{-1}(K)$

similar argument for inverses. //

(B) **Orders:** $|H| < \infty \Longrightarrow \varphi(H) \,\big|\, |H|$

Reason: Restrict φ to H. It's still a homomorphism, $\varphi_H : H \longrightarrow \varphi(H)$

$\varphi(h_1) = \varphi(h_2) \iff h_1 \operatorname{Ker}\varphi = h_2 \operatorname{Ker}\varphi$ (same coset)

$\therefore |\varphi(H)| = \#$ of cosets of $\operatorname{Ker}\varphi$.

$\therefore\ |H| = |\operatorname{Ker}\varphi|\ |\varphi(H)|$. //

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(C) (i) $H \subset G$ normal $\Rightarrow \varphi(H)$ normal in $\varphi(G)$.

(ii) $K \subset \bar{G}$ normal $\Rightarrow \varphi^{-1}(K)$ normal in G.

Check (i) For $a \in G$, need to check

$\varphi(a)\,\varphi(H) = \varphi(H)\,\varphi(a)$. (*)

$x \in \varphi(a)\varphi(H) \Leftrightarrow \exists\ h \in H$ with $x = \varphi(a)\varphi(h)$
$= \varphi(ah)$

Left coset

Since H is normal in G, $\exists\ h' \in H$ s.t. $ah = h'a$ $\therefore \varphi(ah) = \varphi(h'a) = \varphi(h')\varphi(a)$
$\in \varphi(H)\,\varphi(a)$

right coset

Check (ii)!

<u>Note</u> It is <u>not</u> true that $H \subset G$ normal
$\Rightarrow \varphi(H)$ normal in $\bar{G}$.

<u>Ex.</u> Take $G = S_2$, $H = S_2$ (Any group is a normal subgroup of itself.)

$S_2 = \{id, (12)\}$. Take $\bar{G} = S_3$

& $\varphi : S_2 \longrightarrow S_3$ given by

$\varphi(12) = (12) \in S_3$

We know $\{id, (1,2)\}$ <u>not</u> normal in S_3. Here $\varphi(S_2) = S_2 \subset S_3$

permutations of $\{1,2,3\}$

Permutations of $\{1,2\}$ leaving 3 fixed.

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(D) suppose $\varphi: G \to \bar{G}$ is onto, i.e. $\varphi(G) = \bar{G}$
Then

$$\boxed{\varphi \text{ is an isomorphism} \iff \ker\varphi = \{e\}}$$

Important!

Reason: Need to show φ 1-1 $\iff$ $\ker\varphi = \{e\}$

$\Rightarrow$ Assume φ is 1-1 & let $x \in \operatorname{Ker}\varphi$
Then $\varphi(x) = \bar{e} = \varphi(e) \Rightarrow x = e$.

$\Leftarrow$ If $\operatorname{Ker}\varphi = \{e\}$
$a\operatorname{Ker}\varphi = b\operatorname{Ker}\varphi \Rightarrow a = b$

This is (e) on first page. (Recall: $\varphi(a) = \varphi(b) \iff a\operatorname{Ker}\varphi = b\operatorname{Ker}\varphi$)

Important:

Thm. Ker φ is a normal subgroup of G

Pf: By criterion, need to show $\forall a \in G$,
$a(\operatorname{Ker}\varphi)a^{-1} \subset \operatorname{Ker}\varphi$. Let $x \in \operatorname{Ker}\varphi$. Then
$\varphi(axa^{-1}) = \varphi(a)\varphi(x)\varphi(a^{-1}) = \varphi(a)\varphi(a^{-1}) = e$.
$\therefore axa^{-1} \in \operatorname{Ker}\varphi$. //

Ex. Determine all homomorphisms
$$\varphi: \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{30}$$

Ans. Since 1 generates $\mathbb{Z}_{12}$, φ is determine by $\varphi(1)$. Put $a = \varphi(1) \in \mathbb{Z}_{30}$. Are all

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Then $\boxed{\varphi(n) = (a + a + \cdots + a) \bmod 30}$
$= (na) \bmod 30.$

Are all values of $\mathbb{Z}_{30}$ possible for a?
Need $\boxed{\varphi(0)=0}$, otherwise no restriction.
Since $\varphi((n+m) \bmod 12) = \varphi(n) + \varphi(m)$

need $\underset{\underset{2\times2\times3}{\shortparallel}}{12} \mid n+m \Longrightarrow \underset{\underset{2\times3\times5}{\shortparallel}}{30} \mid (n+m)a$ $\quad \forall\, n, m \in \mathbb{Z}_{12}$

For this it is necessary & sufficient that $5 \mid a$. (Take $n=3$, $m=9$.) So

$$\boxed{a \in \{0, 5, 10, 15, 20, 25\}}$$

$\varphi(n) = 0$, or $\varphi(n) = (5n) \bmod 30$ or $\varphi(n) = (10n) \bmod 30$
or $\varphi(n) = (15n) \bmod 30$, or $\varphi(n) = (20n) \bmod 30$
or $\varphi(n) = (25n) \bmod 30$

$\boxed{6 \text{ homomorphisms } \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{30}}$

Orders of a in $\mathbb{Z}_{30}$: $|5| = 6$
$|10| = 3$
$|15| = 2$
$|20| = 3$
$|25| = 6$

$|\varphi(\mathbb{Z}_{12})| = 6$ or 3 or 2

First isomorphism Th

If $\varphi : G \longrightarrow \bar{G}$ homomorphism, then

$$G/_{\ker\varphi} \approx \varphi(G)$$

Note that $G/_{\ker\varphi}$ is a group, since $\ker\varphi$ is normal

Ex 1. $\mathbb{Z}/_{\langle n\rangle} \approx \mathbb{Z}_n$.

Ex 2. $A \oplus B /_{A \oplus \{e\}} \approx B$ by $(a,b) \longmapsto (a,e)$

Need to check that $A \oplus \{e\}$ normal, which is easy. Do it!

Cor. If $\varphi : G \longrightarrow \bar{G}$ homomorphism Then $|\varphi(G)|$ divides both $|G|$ & $|\bar{G}|$

Reason: ① $\varphi(G) \approx G/_{\ker\varphi}$

$$\therefore |\varphi(G)| = |G/_{\ker\varphi}| = \frac{|G|}{|\ker\varphi|}$$

② $\varphi(G)$ is a subgroup of $\bar{G}$

$$\therefore |\varphi(G)| \mid \bar{G}$$