Math 140A– Sample Midterm

Caution: This is not the real midterm. the real midterm will contain problems not on this sample midterm! This sample midterm is intended to give you an idea of what I think a real midterm might look like.

Instructions. Answer all questions. You may use without proof anything which was proved in class or in the text by Rudin (unless the problem explicitly states otherwise). Cite a theorem either by name, if it has one, or by briefly stating what it says. However, you must reprove items which were given as exercises. Unless otherwise stated, X is a metric space and E is a subset of X. E' denotes the set of limit points of E in X, and E^o denotes the interior of E. The notation $\mathbb{R}$ is used for the real numbers, with the usual metric, and $\mathbb{Q}$ for the rational numbers.

1. (15 pts.) Show, by using only the definitions and without citing any theorems, that if $\{p_n\}$ is a convergent sequence in X, it is a Cauchy sequence.

2. (20 pts.) Let $K \subset X$ be compact. Show that K contains a dense subset which is at most countable.

3. (15 pts.) Let $\{a_n\}$ be any sequence of real numbers such that $|a_j| \to \infty$ as $j \to \infty$. Show that the set $\{a_1, a_2, \dots\}$ has no limit points.

4. (10 pts. each part) True or false. For each part, determine if it is always true or sometimes false. If true give a reason or short proof. If false give a counterexample. No credit if reason is missing or incorrect. It's OK to be brief here.

(a) If $K \subset X$ is compact and $F \subset X$ is closed, then $F \cap K$ is compact.

(b) Suppose $\{p_n\}$ is a sequence in X and there is $p \in X$ such that every neighborhood of p contains infinitely many distinct points of the set $\{p_1, p_2, \dots\}$. Then $p = \lim p_n$.

(c) Suppose $E \subset \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^4 + z^6 < 5\}$. If E contains all its limit points, then E is compact.

(d) Let a_n, $n = 1, 2, \dots$, be real numbers satisfying $|a_n| \leq n^2+n+1$ and $\frac{a_n}{n^2} \neq \frac{a_m}{m^2}$ if $n \neq m$. Then the set $\{\frac{a_1}{1^2}, \frac{a_2}{2^2}, \dots \frac{a_n}{n^2}, \dots\}$ has at least one limit point.

(e) Suppose that X is a countable set. Let $\mathcal{S}$ be the set of all finite subsets of X, i.e.

$$\mathcal{S} = \{E \subset X : E \text{ has finitely many elements}\}.$$

Then $\mathcal{S}$ is countable.