NOTES ON RANDOM WALKS

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1. Simple Random Walks

Let ξ be a random variable taking integer values, and $x \in \mathbb{Z}$. For $i = 1, 2, \ldots$ we successively walk ξ units right or left, depending on the sign of ξ, from an initial position x. We define a sequence of random variables X_n, expressing the position at time n by

$$X_0 = x, X_1 = x + \xi_1, \ldots, X_n = x + \xi_1 + \xi_2 + \ldots + \xi_n, \tag{1}$$

where the ξ_i are independent, identically distributed with distribution the same as that of ξ, with finite mean μ and finite variance σ^2. The sequence $\{X_n\}$ is called a *random walk*. If ξ takes only the values $0, -1, +1$, the sequence is a *simple random walk*. By the additivity of expectation of independent random variables,

$$E[X_n] = x + nE[\xi]. \tag{2}$$

Now let T be a random variable taking positive integer values, with finite mean $E[T]$, independent of the ξ_i for $i > T$. We think of T as a stopping time, and are interested in the random variable X_T (which is a *random sum*).

1.1. Wald's identity.

. The expected value $E[X_T]$ and variance $\mathrm{Var}[X_T]$ satisfy the following identities due to Wald.

$$E[X_T] = x + E[\xi]E[T] \tag{3}$$

If, in addition, $E[\xi] = 0$, then

$$\mathrm{Var}[X_T] = \sigma^2 E[T], \tag{4}$$

where $\sigma^2 = Var(\xi_i)$. Idea of proof:

$$E[X_T] = \sum_{n=1}^{\infty} E[X_T | T = n] \mathrm{P}\{T = n\} \tag{5}$$

$$= \sum_{n=1}^{\infty} E[X_n] \mathrm{P}\{T = n\} \quad \text{(by independence of } \xi_i \text{ and } T \text{ for } i > T) \tag{6}$$

$$= \sum_{n=1}^{\infty} (x + nE[\xi]) \mathrm{P}\{T = n\} \tag{7}$$

$$= x + E[\xi] \sum_{n=1}^{\infty} n \mathrm{P}\{T = n\}, \tag{8}$$

which proves (3). For (4), assume $E[\xi] = 0$ and write

$$\mathrm{Var}(X_T) = E[(\sum_{i=1}^{T} \xi_i - E[\xi]E[T])^2] = E[(\sum_{i=1}^{T} \xi_i)^2] \tag{9}$$

By the law of total probability,

$$E[(\sum_{i=1}^{T} \xi_i)^2] = \sum_{n=1}^{\infty} E[(\sum_{i=1}^{T} \xi_i)^2 | T = n] \cdot \mathrm{P}\{T = n\} = \sum_{n=1}^{\infty} E[(\sum_{i=1}^{n} \xi_i)^2] \cdot \mathrm{P}\{T = n\} \tag{10}$$

By independence of the ξ_j, and the assumption $E[\xi_j] = 0$,

$$E[\xi_j \xi_k] = 0 \quad \text{for} \quad j \neq k. \tag{11}$$

Then

$$\mathrm{Var}(X_T) = \sum_{n=1}^{\infty} (\sum_{j=1}^{n} E[\xi_j^2]) \cdot \mathrm{P}\{T = n\} = \tag{12}$$

$$= \mathrm{Var}[\xi] (\sum_{n=1}^{\infty} n \mathrm{P}\{T = n\}) = \mathrm{Var}[\xi] E[T]$$

1.2. **Applications of Wald's identity to simple random walks (gambler's ruin).** In the following examples, players Jack and Jill will start with \$5 and \$10 respectively and play a game by making a series of \$1 bets until one of them loses all his/her money. We'll assume that in each bet, Jack wins with probability p, Jill with probability q, and tie (no money exchanged) with probability r, so that

$$p + q + r = 1.$$

To model this as a simple random walk, we let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. random variables taking the values $1, -1, 0$ with probabilities p, q, r respectively. Let X_n be the amount of money that Jack has after n bets, so that

$$X_n = 5 + \xi_1 + \xi_2 + \ldots + \xi_n.$$

Note that by the rules of the game, $0 \leq X_n \leq 15$ for all n. Let T be the number of bets made until the game ends. Then T is a stopping time, and X_T represents the amount of money that Jack has when the game ends. The random sum X_T and take only two values, 0 (Jill wins) or 15 (Jack wins). We will use Wald's identities to calculate $E[T]$ and the respective probabilities of Jack or Jill winning.

Example 1.1. Suppose that $p = q = 1/2$, $r = 0$, so that $E[\xi] = 0$. By the first Wald identity (3), it follows that

$$E[X_T] = 5.$$

Then

$$E[X_T] = 0 \cdot \mathrm{P}(X_T = 0) + 15 \cdot \mathrm{P}(X_T = 15) = 5, \tag{13}$$

so that

$$\mathrm{P}(X_T = 15) = 1/3,$$

i.e.

$$P(\text{Jack wins}) = 1/3,$$

while

$$P(\text{Jill wins}) = 2/3.$$

This also gives

(14)

$$\begin{aligned}\mathrm{Var}[X_T] = E[X_T^2] - 5^2 = 0 \cdot \mathrm{P}(X_T^2 = 0) + 15^2 \cdot \mathrm{P}(X_T^2 = 15^2) - 25 \\ = 15^2 \cdot 1/3 - 25 = 50.\end{aligned}$$

By the second Wald identity (4), $\mathrm{Var}[X_T] = E[\xi^2]E[T]$. Also,

$$\xi^2 \equiv 1 \implies E[\xi^2] = 1.$$

Hence $E[T] = 50$.

Example 1.2. Now suppose $p \neq q$, so that (4) does not hold. We set up an inductive process for $0 \leq x \leq 15$ by putting

$$X_n(x) = x + \sum_{j=1}^{n} \xi_j, \quad X_0 = x, \quad \text{and let } \mathrm{f(x) := P(X_T(x) = 15)} \tag{15}$$

Then

$$f(0) = 0, \quad f(15) = 1, \tag{16}$$

and we are interested in computing $f(5)$. Conditioning on the possible values of ξ_1, we have

$$f(x) = f(x+1)\mathrm{P}(\xi_1 = 1) + f(x-1)\mathrm{P}(\xi_1 = -1) = p \cdot f(x+1) + q \cdot f(x-1). \tag{17}$$

Writing

$$f(x) = pf(x) + qf(x),$$

we obtain from (17),

$$f(x) - f(x-1) = (p/q)(f(x+1) - f(x)) \tag{18}$$

Now define

$$h(x) := f(x) - f(x-1),$$

and note that $h(1) = f(1)$. The collapsing sum gives

$$f(x) = \sum_{j=1}^{x} h(j) + f(0) = \sum_{j=1}^{x} h(j). \tag{19}$$

On the other hand, since

$$h(x+1) = (q/p)h(x)$$

by (18), we have by induction,

$$h(x) = (q/p)^{x-1}h(1) = (q/p)^{x-1}f(1).$$

From (16) and (19), we then have

$$1 = f(15) = \sum_{j=1}^{15} h(j) = \sum_{j=1}^{15} (q/p)^{j-1} f(1),$$

which gives

$$f(1) = \frac{1}{\sum_{j=1}^{15} (q/p)^{j-1}}$$

By putting this into (19), we may calculate the desired $f(5)$ explicitly by

$$f(5) = \sum_{j=1}^{5} h(x)f(1) = \sum_{j=1}^{5} (q/p)^{j-1} f(1) = \frac{\sum_{j=1}^{5} (q/p)^{j-1}}{\sum_{j=1}^{15} (q/p)^{j-1}} = \frac{1-(q/p)^5}{1-(q/p)^{15}} \tag{20}$$

To find $E[T]$, we begin by finding $E[X_T]$ in order to apply (3).

$$E[X_T] = 0 \cdot [\mathrm{P}\{X_T = 0\}] + 15f(5) = 15\frac{1-(q/p)^5}{1-(q/p)^{15}} \tag{21}$$

Then

$$E[T] = (E[X_T] - 5)/E[\xi] = \frac{15\frac{1-(q/p)^5}{1-(q/p)^{15}} - 5}{p - q}. \tag{22}$$