Midterm II Solutions

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1) $s = .5 + .4s + .1s^2$.

2)

$$P = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a) $w_3 = 0$, $w_2 = 1 + (1/2)w_2$, $w_1 = 1 + (1/3)w_1 + (1/3)w_2$, $w_0 = 1 + (1/2)w_0$.

b) $w_3 = 0$, $w_2 = 2$, $w_1 = 5/2$.

3) For $i = 1, 2, 3$ let $q_i = P\big(X_{T-1} = 2 | X_0 = i\big)$. Then $q_3 = 0$, $q_2 = 1$ and $q_1 = (1/3)q_1 + 1/3$. The last equation gives that $q_1 = 1/2$.

4)

$$P = \begin{pmatrix} .5 & .4 & .1 & 0 & 0 \\ 0 & .5 & .4 & .1 & 0 \\ 0 & 0 & .5 & .4 & .1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

a) $q = (.5)q_0 + (.4)q_1 + (.1)q_2$

b) $q_0 = (.5)q_0 + (.4)q_1 + (.1)q_2$, $q_1 = (.5)q_1 + (.4)q_2$, $q_2 = (.5)q_2 + (.1)$

5) a) $f_{00}^1 = a_{00}$ and for $n \geq 1$ $f_{00}^n = 0$.

b) $a_{00} < 1$.

6) i) The Markov chain is irreducible. State 0 communicates with any other state j since $P_{0j} = \beta_{j+1}$ and once in state j Markov chain deterministically returns to state 0 in **j** steps. This means that all states communicate and hence Markov chain is irreducible.
ii) The Markov chain is aperiodic: $P_{00} > 0$ therefore $d(0) = 1$. Period is a class property and since there is a single communicating class $d(j) = 1$ for all j. Hence the Markov chain is aperiodic.
iii) Zero is a recurrent state:
$f_{00}^n = P\big(X_n = 0, X_{n-1} \neq 0, \ldots, X_1 \neq 0 | X_0 = 0\big) = \beta_n$, since it takes one step to get to state $n-1$ and $\mathbf{n-1}$ steps to get from state $n-1$ to 0. We get that $f_{00} = \sum_{n=1}^{\infty} f_{00}^n = 1$, hence 0 is a recurrent state. Recurrence is a class property, hence the Markov chain is recurrent.
The three conditions above imply that $\lim_{n \longrightarrow \infty} P_{ii}^n$ exists for all i.