Here is a list of problems to prepare the final.

**Problem 1.** Prove the following propositions:

1. \( \neg(P \lor Q) \) is equivalent to \((\neg P) \land (\neg Q)\).
2. \( P \Rightarrow Q \) is equivalent to \((P \lor Q) \Rightarrow Q\).
3. \( P \land (Q \lor R) \) is equivalent to \((P \land Q) \lor (P \land R)\).

**Problem 2.** Prove by induction that \( \sum_{i=1}^{n} i \cdot i! = (n+1)! - 1 \), for all positive integers \( n \geq 1 \).

**Problem 3.** For a positive integer \( n \) the number \( a_n \) is defined inductively by

\[
a_1 = 3, \quad a_2 = 15, \quad a_n = 5a_{n-1} - 4a_{n-2} \text{ for } n \geq 2.
\]

1. Prove that \( a_n = 4^n - 1 \), for every positive integer \( n \).
2. Prove that \( 3 \mid a_n \), for every positive integer \( n \).

**Problem 4.** For all sets \( A \) and \( B \), prove that \((A \cup B) \cap (A \cap B)^c = (A \setminus B) \cup (B \setminus A)\).

**Problem 5.** Prove or disprove the following statements.

1. \( \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, |x - y| < 1; \)
2. \( \forall n \in \mathbb{Z}, (\exists p, q \in \mathbb{Z} \ (n = 3p \Rightarrow n = 6q)) \).

**Problem 6.** Define the functions \( f, g : \mathbb{R} \to \mathbb{R} \) by \( f(x) = x^2 \) and \( g(x) = x^2 - 1 \).

1. Find the functions \( f \circ f, f \circ g, g \circ f, \) and \( g \circ g \).
2. Suppose we changed the codomain of \( f \) to be \( \mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\} \). Which of the functions in part (a) would still be well defined?
3. List all elements of the set \( \{x \in \mathbb{R} : f(g(x)) = g(f(x))\} \).

**Problem 7.** Let \( f : X \to Y \) be a function and let \( G_f \subset X \times Y \) be the graph of \( f \). Prove that \( \text{Im}(f) = \{y \in Y : (X \times \{y\}) \cap G_f \neq \emptyset\} \).

**Problem 8.** For \( n \in \mathbb{N} \), suppose that \( A \subset \mathbb{N}_{2n} \) and \( |A| = n + 1 \). Prove that \( A \) contains a pair of distinct integers \( a, b \) such that \( a \) divides \( b \). (Hint: for each element of \( A \), consider the largest odd number dividing it. You can freely use the fact that if \( n \) is an integer that is not divisible by any odd number greater than 1 then there is an integer \( k \geq 1 \) so that \( n = 2^k \).)
Problem 9. Let $n$ be a positive integer and $1 \leq k \leq n$. Prove that
\[
\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}.
\]

Problem 10. Let $p, q \in \mathbb{Z}$ such that $3$ divides $p^2 + q^2$. Use the division theorem to prove that $3$ divides $p$ and $3$ divides $q$.

Problem 11 (Bonus). Prove that among any five points selected inside an equilateral triangle with side equal to one inch, there always exists a pair of points at distance not greater than one half an inch.