MATH 109 MIDTERM 2

Wednesday, November 20, 2019 (50 minutes).

Please turn all cell phones off completely and put them away.
No books, notes, or electronic devices are permitted during this exam.
Generally, you must show your work to receive credit.

- The back side of test pages will not be evaluated. You can use it for your own computations, but whatever you write there will not be evaluated.

Name (print): ____________________________________________

Student ID number: ______________________________________

<table>
<thead>
<tr>
<th>Question</th>
<th>Parts</th>
<th>Points</th>
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</thead>
<tbody>
<tr>
<td>Question 1</td>
<td>one part</td>
<td>25 points</td>
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<tr>
<td>Question 2</td>
<td>two parts</td>
<td>10+15=25 points</td>
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<tr>
<td>Question 3</td>
<td>two parts</td>
<td>15+10=25 points</td>
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<td>Question 4</td>
<td>one part</td>
<td>25 points</td>
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<tr>
<td>Bonus Question</td>
<td>two parts</td>
<td>7+8=15 points</td>
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<tr>
<td>Total</td>
<td>4 questions</td>
<td>100 points</td>
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THIS SIDE WILL NOT BE EVALUATED!
1. [25 pts] Let $A, B$ be two sets. Prove that

$$(A \setminus B) \cap (B \setminus A) = \emptyset$$

**Proof 1:** Suppose $(A \setminus B) \cap (B \setminus A) \neq \emptyset$ by contradiction,

then $\exists x_0 \in (A \setminus B) \cap (B \setminus A) \Rightarrow x_0 \in A \setminus B \cap x_0 \in B \setminus A

\Rightarrow x_0 \notin A \land x_0 \notin B \land x_0 \notin A \land x_0 \notin B \land \ldots$  

\text{No}

**Proof 2:** Table of Truth

<table>
<thead>
<tr>
<th>$x \in A$</th>
<th>$x \in B$</th>
<th>$x \in A \cap B$</th>
<th>$x \in B \setminus A$</th>
<th>$x \in (A \cap B) \setminus (B \setminus A)$</th>
<th>$x \notin \Phi$</th>
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Note: Dem's diagram is NOT a proof.
THIS SIDE WILL NOT BE EVALUATED!
2. [25 pts] Prove or disprove the following statements:

2.1 [10 pts] \( \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y - x \geq 0. \)

\[ \text{The negation is } \forall x \in \mathbb{R} \exists y \in \mathbb{R}, y - x < 0, \text{ and this is true, indeed given } x \in \mathbb{R} \text{ choose } y = x + 1, \text{ then} \]
\[ y - x = x + 1 - x = 1 > 0. \]

2.2 [15 pts] \( (\exists a \in \mathbb{Z}, n = 2a + 1) \Rightarrow (\exists b \in \mathbb{Z}, n^2 = 4b + 1). \)

\[ \text{Proof: Direct proof: Suppose } \exists a \in \mathbb{Z} \text{ and } n = 2a + 1 \]
\[ n^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 4(a^2 + a) + 1 \]
\[ \text{Then } n^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 4(a^2 + a) + 1 \]
\[ b \in \mathbb{Z} \]
THIS SIDE WILL NOT BE EVALUATED!
3. [25 pts] Let \( f: X \rightarrow Y \) and \( g: Y \rightarrow Z \).
3.1 [15 pts] Prove that if \( g \circ f \) is injective, then \( f \) is injective;

\[
\text{Proof: Suppose } f(x) = f(y), \text{ then apply } g \text{ on both sides to get } \]
\[
g \circ f(x) = g \circ f(y) \Rightarrow g(f(x)) = g(f(y)). \text{ Since } g \circ f \text{ is injective } \Rightarrow x = y, \text{ that is } f \text{ is injective.} \]

3.2 [10 pts] Is it true that if \( g \circ f \) is injective, then \( g \) is injective? If yes prove it, if not find a counterexample.

It’s not true, indeed let \( X = \{1\} \), \( Y = \{1, 2\} \) and \( Z = \{1\} \)
and consider the functions

\[
f: X \rightarrow Y, \quad g: Y \rightarrow Z \quad \text{so that}
\]
\[
1 \mapsto f(1) = 1, \quad 1, 2 \mapsto g(1) = 1, 2 \mapsto g(2)
\]

Then \( g \circ f (x) = g \circ f(y) \Rightarrow x = 1 = y \Rightarrow g \circ f \) injective, \( \text{but } g(1) = g(2) = 1 \)
\( \Rightarrow g \) is not injective.
THIS SIDE WILL NOT BE EVALUATED!
4. [25 pts] A large software development company employs 100 computer programmers. Of them, 45 are proficient in Java, 30 in C, 20 in Python, 6 in C and Java, 1 in Java and Python, 5 in C and Python, and just 1 programmer is proficient in all three languages above.

Determine the number of computer programmers that are not proficient in any of these three languages.

\[
\begin{align*}
\mathcal{S} &= \{\text{Java programmers}\} \\
\mathcal{P} &= \{\text{Python programmers}\} \\
\mathcal{C} &= \{\text{C programmers}\} \\
|\mathcal{S}| &= 45 \\
|\mathcal{P}| &= 30 \\
|\mathcal{C}| &= 20 \\
|\mathcal{S} \cap \mathcal{P}| &= 6 \\
|\mathcal{S} \cap \mathcal{C}| &= 1 \\
|\mathcal{C} \cap \mathcal{P}| &= 5 \\
|\mathcal{S} \cap \mathcal{P} \cap \mathcal{C}| &= 1 \\
\end{align*}
\]

The solution is

\[
100 - |\mathcal{S} \cup \mathcal{P} \cup \mathcal{C}| = 100 - (45 + 30 + 20 - 6 - 5 - 1 + 1) = 100 - 84 = 16
\]
THIS SIDE WILL NOT BE EVALUATED!
5. [15 pts] (Bonus Problem) Prove the following statements.

5.1 [7 pts] If \( X \subset Y \) and \( Y \) is a finite set, then \( X \) is a finite set and \(|X| \leq |Y|\).

Proof: \( Y \) is a finite set, then \( \exists m \in \mathbb{N} \) s.t. \(|Y| = m\), i.e.

\[ \exists f: N_m \rightarrow Y \text{ injection}. \]

Let \( i: X \rightarrow Y \) the inclusion map \( x \mapsto i(x) = x \).

Then \( i \) is injective \( i(x) = i(y) \Rightarrow x = y \) by definition, and \( \exists \)

\[ i \circ f: X \rightarrow N_m \text{ is injective. By pigeonhole principle} \]

\[ |X| \leq m = |Y|. \]

\[ \Box \]

5.2 [8 pts] If \( X, Y \subset \mathbb{N}_n \) are finite sets and \(|X| + |Y| > n\), then \( X \cap Y \neq \emptyset \).

Proof: \( X, Y \subset \mathbb{N}_n \Rightarrow X \cup Y \subset \mathbb{N}_m \Rightarrow \) by 5.1

\[ |X \cup Y| \leq m. \text{ By inclusion exclusion principle} \]

\[ m \geq |X \cup Y| = |X| + |Y| - |X \cap Y| > m - |X \cap Y| \]

\[ \Rightarrow |X \cap Y| > m - m = 0 \Rightarrow |X \cap Y| > 0 \Rightarrow X \cap Y \neq \emptyset. \]

\[ \Box \]
THIS SIDE WILL NOT BE EVALUATED!