Problem 2. Prove by induction. Let $P(n)$ be the statement $\sum_{i=1}^{n} u_i^2 = u_n + \frac{n}{2} u_n$. 

\begin{align*}
\text{Base case: } \sum_{i=1}^{1} u_i^2 &= u_1^2 = 1 \quad \text{and} \quad u_2 u_1 = 1 \cdot 1 = 1, \quad \text{so } P(1) \text{ holds.} \\
\text{Inductive step: } \sum_{i=1}^{n} u_i^2 &= \sum_{i=1}^{n-1} u_i^2 + u_n^2 & \text{by definition} \\
\text{Assume } P(n) \text{ holds for some } n \geq 1, \\
\text{then } u_{n+1} u_n + u_n^2 &\quad \text{by induction hypothesis} \\
&= (u_n + u_{n+1}) u_{n+1} & \text{(i.e. } P(n) \text{ is true)} \\
&= u_{n+2} u_{n+1} & \text{by definition of Fibonacci sequence}
\end{align*}

So the proof is completed. $\star$

Problem 4. (i) Let $P(n)$ be the statement $a_n > 0$.

\begin{align*}
\text{Base case: } a_1 &= 1 > 0, \quad \text{so } P(1) \text{ holds.} \\
\text{Inductive step: } P(n) \text{ is true } \Rightarrow a_n > 0 \\
\text{Assume } P(n) \text{ holds for some } n \geq 1, \\
\Rightarrow 6a_n + 5 > 0 \quad \text{and} \quad a_{n+2} > 0 \\
\Rightarrow a_{n+1} &= \frac{6a_n + 5}{a_{n+2}} > 0 \\
\Rightarrow P(n+1) \text{ is true.}
\end{align*}

Hence the proof is completed. $\star$

(ii) Let $Q(n)$ be the statement $a_n < 5$.

\begin{align*}
\text{Base case: } a_1 &= 1 < 5, \quad \text{so } Q(1) \text{ is true.} \\
\text{Inductive step: observe that } a_{n+1} &= \frac{6a_n + 5}{a_{n+2}} = 6 - \frac{7}{a_{n+2}}. \quad \text{Therefore,} \\
Q(n+1) \text{ is true} \iff a_{n+1} < 5 \iff 6 - \frac{7}{a_{n+2}} < 5 \\
\iff \frac{7}{a_{n+2}} > 1 \\
\iff a_{n+2} < 7 \quad \text{since } a_{n+2} > 0 \text{ as } a_n > 0 \text{ by part (i)} \\
\iff a_n < 5 \\
\iff Q(n) \text{ is true.}$
\end{align*}
Problem 5. We proceed by strong induction. Let $P(n)$ be the statement $T_n < 2^n$.

Base case: $T_1 = 1 < 2^1 \implies P(1)$ holds.

$T_2 = 1 < 2^2 \implies P(2)$ holds

$T_3 = 1 < 2^3 \implies P(3)$ holds

Inductive step: Suppose $P(n-1)$, $P(n-2)$, and $P(n-3)$ are true for some integer $n \geq 4$. Then

$T_n = T_{n-1} + T_{n-2} + T_{n-3}$ by definition of Tribonacci sequence

$< 2^{n-1} + 2^{n-2} + 2^{n-3}$ since $P(n-1), P(n-2),$ and $P(n-3)$ are true

$< 2^{n-1} + 2^{n-2} + 2^{n-2}$ since $2^{n-3} < 2^{n-2}$ (for $n \geq 4$)

$= 2^{n-1} + 2 \cdot 2^{n-2}$

$= 2^{n-1} + 2^{n-1}$

$= 2 \cdot 2^{n-1}$

$= 2^n$

ie. $T_n < 2^n$, as desired.