Final Practice 2.

Problem 1 (1) By truth table: $P \land \neg (P \lor \neg Q)$

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$- (P \lor \neg Q)$</th>
<th>$-P \land \neg Q$</th>
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Problem 2 (2) By truth table: $P \land (P \lor Q) \Rightarrow Q$, $P \Rightarrow Q$

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<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \lor Q$</th>
<th>$(P \lor Q) \Rightarrow Q$</th>
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(3) By truth table:

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<th>$R$</th>
<th>$P \land Q$</th>
<th>$P \land R$</th>
<th>$(P \land Q) \lor (P \land R)$</th>
<th>$Q \lor R$</th>
<th>$P \land (Q \lor R)$</th>
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Problem 2. Let \( P(n) \) be the statement \( \sum_{i=1}^{n} i \cdot i! = (n+1)! - 1 \).

Base case: for \( n = 1 \), \( \sum_{i=1}^{1} i \cdot i! = 1 \cdot 1! = 1 \) and \( (1+1)! - 1 = 2! - 1 = 1 \), hence \( P(1) \) is true.

Inductive step: suppose \( P(n) \) is true for some positive integer \( n \).

Then \( \sum_{i=1}^{n+1} i \cdot i! = \sum_{i=1}^{n} i \cdot i! + (n+1)! \cdot (n+1)! \) by definition of \( \Sigma \)

\[ = (n+1)! - 1 + (n+1)! \cdot (n+1)! \text{ by induction hypothesis} \]

\[ = (1 + (n+1)) \cdot (n+1)! - 1 \]

\[ = (n+2) \cdot (n+1)! - 1 \]

\[ = (n+2)! - 1 \]

Hence \( P(n+1) \) is true, as desired.

Problem 3 (i). Let \( P(n) \) be the statement \( a_n = 4^n - 1 \).

Base case: for \( n = 1 \), \( a_1 = 3 \) while \( 4^1 - 1 = 3 \), hence \( P(1) \) is true.

for \( n = 2 \), \( a_2 = 15 \) while \( 4^2 - 1 = 15 \), hence \( P(2) \) is true.

Inductive step: suppose \( P(n) \) and \( P(n+1) \) are true for some positive integer \( n \). Then

\[ a_{n+2} = 5a_{n+1} - 4a_n \quad \text{by definition of } a_n \]

\[ = 5 \cdot (4^{n+1} - 1) - 4(4^n - 1) \quad \text{by strong induction hypothesis} \]

\[ = 5 \cdot 4^{n+1} - 5 - 4 \cdot 4^n - 4 \]

\[ = (5 \cdot 4^n - 1) - 4 - 1 \]

\[ = (20 - 4) 4^n - 1 \]

\[ = 4^{2n} - 1 \]

Hence \( P(n+2) \) is true, as desired.
Problem 3 (2) We know by part (1) that \( a_n = 4^n - 1 \) for every positive integer \( n \), therefore, it is enough to prove \( 3 | 4^n - 1 \) for every positive integer \( n \).

Let \( P(n) \) be the statement \( 3 | 4^n - 1 \).

Base case: for \( n = 1 \), \( 4^1 - 1 = 3 \), so \( 3 | 4^1 - 1 \), i.e. \( P(1) \) is true.

Inductive step: Suppose \( P(n) \) is true for some positive integer \( n \). Then,

\[
4^{n+1} - 1 = 4(4^n - 1)
\]

\[
= 4(4^n) - 1
= 4(4^n - 1) + 4 - 1
= 4(3k) + 3
\]

for some integer \( k \), by induction hypothesis.

\[
= 3(4k + 1)
\]

where \( 4k + 1 \) is an integer (since \( k \) is an integer).

Hence \( 3 | 4^{n+1} - 1 \), as desired.

\[
\]

Problem 4. By truth table:

<table>
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<tr>
<th>( x \in A )</th>
<th>( x \in B )</th>
<th>( A \cup B )</th>
<th>( (A \cap B)^c )</th>
<th>( (A \cup B)^c \cap (A \cap B)^c )</th>
<th>( (A \setminus B) \cup (B \setminus A) )</th>
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Hence \( (A \cup B)^c \cap (A \cap B)^c = (A \setminus B) \cup (B \setminus A) \).

Problem 5 (1) Disprove by counterexample: for any \( x \in \mathbb{R} \), let \( y = x - 1 \). Then \( y \in \mathbb{R} \) as \( x - 1 \in \mathbb{R} \), and \( |x - y| = |x - (x - 1)| = 1 \) is not less than 1.
Problem 5 (2) Pf: Given \( n \in \mathbb{Z} \), we can let \( p = n^2 + 1 \in \mathbb{Z} \) and \( q = 0 \in \mathbb{Z} \).

Then the statement \( (n = 3p \Rightarrow n = 6q) \) is true since
\[ n = 3p \text{ is a false statement (} n \in \mathbb{Z} \Rightarrow n^2 = n \Rightarrow n^2 + 1 > n \Rightarrow 3p = 3(n^2 + 1) > n. \text{ Hence } n \neq 3p \text{ for each } n \in \mathbb{Z}. \]

Problem 6 (1) \[ f \circ f(x) = f(x^2) = (x^2)^2 = x^4. \]
\[ f \circ g(x) = f(x^2 - 1) = (x^2 - 1)^2. \]
\[ g \circ f(x) = g(x^2) = (x^2)^2 - 1 = x^4 - 1. \]
\[ g \circ g(x) = g(x^2 - 1) = (x^2 - 1)^2 - 1. \]

(2) Since \( f(x) = x^2 \geq 0 \) for all \( x \in \mathbb{R} \), \( \text{Im}(f) \subseteq \mathbb{R}^+ \) by definition.

Therefore, changing the codomain of \( f \) to be \( \mathbb{R}^+ \) actually changes nothing. Hence all the functions in part (a) would still be well defined.

(3) For \( x \in \mathbb{R} \), \( f(g(x)) = g(f(x)) \iff (x^2 - 1)^2 = x^4 - 1 \iff x^4 - x^2 + 1 = x^4 - 1 \iff -2x^2 = -2 \iff x = \pm 1. \)

Hence \( \{x \in \mathbb{R} : f(g(x)) = g(f(x))\} = \{1, -1\} \).

Problem 7 Prove \( \text{Im}(f) = \{y : (x \in X) \land G_f \neq \emptyset \} =: \text{RHS}. \)

pf \( (\subseteq) \) Let \( y \in \text{Im}(f) \). Then by definition \( y = f(x) \) for some \( x \in X \).

So \( (x, y) = (x, f(x)) \in G_f \), while \( (x, y) \in X \times \{y\} \) by definition.

Thus \( (x, y) \in (X \times \{y\}) \cap G_f \), so \( (x \times \{y\}) \cap G_f \neq \emptyset \) and hence \( y \in \text{RHS} \).

\( (\supseteq) \) Let \( y \in \text{RHS} \). Then \( (x \times \{y\}) \neq \emptyset \), and so \( \exists (x, y) \in (X \times \{y\}) \cap G_f \).

In particular, such \( (x, y) \in G_f \), and so \( (x, y) = (x, f(x)) \) by definition of \( G_f \). Therefore \( y = f(x) \), i.e. \( y \in \text{Im}(f) \).

Picture:

![Diagram](attachment:pic.png)
Problem 8. By the hint, consider the following function

\[ A \rightarrow \{ \text{odd numbers in } \mathbb{N}_{2n} \} = \{1, 3, 5, \ldots, 2n-1\} \]

\[ m \rightarrow \text{the largest odd number dividing } m. \]

Since \(|A| = n+1 > n = |\{ \text{odd numbers in } \mathbb{N}_{2n} \}|\), by Pigeonhole principle there exist \(a, b\) in \(A\), \(a \neq b\), such that the largest odd number dividing \(a\) and that of \(b\) are the same, call it \(c\). Then \(\frac{a}{c}\) is an integer that is not divisible by any odd number greater than 1, so by the hint, \(\frac{a}{c} = 2^k\) for some integer \(k\). Similarly, \(\frac{b}{c} = 2^l\) for some integer \(l\). If \(k > l\), then \(\frac{b}{c} = (\frac{b}{c}) \cdot 2^{k-l}\), so \(a = b \cdot 2^{k-l}\) and \(2^{k-l} \in \mathbb{Z}\), and hence \(b\) divides \(a\). Otherwise \(k = l\), then \(\frac{b}{c} = (\frac{b}{c}) \cdot 2^{l-k}\), so \(b = a \cdot 2^{l-k}\) and \(2^{l-k} \in \mathbb{Z}\), and hence \(a\) divides \(b\). In any case, we've found \(a \neq b\) in \(A\) such that one divides the other, as desired. \(\ast\)

Problem 9. By Theorem 12.2.60 (p. 151 in text), we have

\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

\[ = \frac{n \cdot (n-1)!}{k \cdot (k-1)! \cdot (n-k)!} \]

\[ = \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)! \cdot ((n-1)-(k-1))!} \]

\[ = \frac{n}{k} \binom{n-1}{k-1} \]

\(\ast\)
Problem 10. By the division theorem, \( \exists m, r \in \mathbb{Z} \) st. \( p = 3m_1 + r \) with \( 0 \leq r < 3 \), and \( \exists m_2, r_2 \in \mathbb{Z} \) st. \( q = 3m_2 + r_2 \) with \( 0 \leq r_2 < 3 \).

Then \( p^2 + q^2 = (3m_1 + r)^2 + (3m_2 + r_2)^2 \)
\[ = (9m_1^2 + 6m_1r + r^2) + (9m_2^2 + 6m_2r_2 + r_2^2) \]
\[ = 3(3m_1^2 + 2m_1r + 3m_2^2 + 2m_2r_2) + (r^2 + r_2^2), \]
while by assumption \( p^2 + q^2 = 3k \) for some \( k \in \mathbb{Z} \). Therefore
\[ r^2 + r_2^2 = 3k - 3(3m_1^2 + 2m_1r + 3m_2^2 + 2m_2r_2) \]
\[ = 3(k - 3m_1^2 - 2m_1r - 3m_2^2 - 2m_2r_2) \]

an integer.

Hence \( 3 \mid r^2 + r_2^2 \). Notice that \( r, r_2 \) are integers with \( 0 \leq r, r_2 \leq 2 \), let's check the case for \( 3 \mid r^2 + r_2^2 \) to hold:
\[ (r, r_2) = (0, 0) \Rightarrow 3 \mid 0^2 + 0^2 = 0 \] is true.
\[ = (0, 1) \text{ or } (1, 0) \Rightarrow 3 \nmid 1 \]
\[ = (0, 2) \text{ or } (2, 0) \Rightarrow 3 \nmid 4 \]
\[ = (1, 1) \Rightarrow 3 \mid 2 \]
\[ = (1, 2) \text{ or } (2, 1) \Rightarrow 3 \nmid 5 \]
\[ = (2, 2) \Rightarrow 3 \nmid 8 \]

Thus \( r = 0 \) & \( r_2 = 0 \) is the only case for \( 3 \mid r^2 + r_2^2 \) to be true.
Hence \( p = 3m_1 (m, \in \mathbb{Z}) \) and \( q = 3m_2 (m, \in \mathbb{Z}), \) so \( 3 \mid p \) & \( 3 \mid q \).

Problem 11. See the solution to problem 5 in Practice Midterm 2.