Chapter 1

1 Mathematical statements and Propositions

The majority of this class will be dedicated to learning the language of mathematics.

A mathematical sentence is usually called a **statement**.

A **proposition** is a statement which is either true or false.

**Examples:**

1. \(1 + 1 = 2\)
2. \(7 = 3\)
3. Every even integer greater than 2 may be written as the sum of two prime numbers.
4. \(12 - 11\)
5. \(a \cdot b = 0\)

\(\circ\) is true, \(\odot\) is false, \(\ominus\) is very hard (Goldbach Conjecture)

\(\ominus\) is not a proposition (true or false)

\(\odot\) is a proposition if you assign values to \(a\) and \(b\).
ex. Q: Are the following propositions?
① \( m^2 - 2m > 0 \) or \( m < 0 \), but \( m = 3 \) then \( m^2 - 2m > 0 \) \( \Box \)
② \( m^2 - 2m + 1 > 0 \) or confusing, it is true \( m^2 - 2m + 1 = (m-1)^2 \) for every value of \( m \).

Notation: We will denote propositions with capital letters \( P, Q, \ldots \)
- \( P, Q, \ldots \) (example: \( 3 > 2 \) might be called \( P \))
- \( P(m, m), Q(x, y) \) propositions depending on \( m, m, x, y \)
  (example: \( P(m, m) \Rightarrow m+m > 2 \))

\( m, m, x, y \) are sometimes called free-variables. \( P(m, m) \) becomes a proposition when \( m, m \) are specified.

① Connections: or, and, not

These are operations we can perform to combine simple propositions into complicated ones.

To decide whether the new complicated proposition is true, knowing if the simple ones are true or false, we use the so-called truth tables.
\[\begin{array}{c|c|c}
T & T & T \\
F & T & T \\
F & T & F \\
F & F & F \\
\end{array}\]

Example 1: \(P \approx 2 \leq 3\)
\(Q \approx 2 = 3\) then \(P \lor Q \approx 2 \leq 3\)

Maybe it is better (and more useful) to think backward

"2 \leq 3 is the proportion 2 \leq 3 or 2 = 3"

So 2 \leq 3 true? \(\begin{array}{c|c|c}
2 \leq 3 & 2 = 3 & 2 \leq 3 \lor 2 = 3 \\
T & F & T \\
\end{array}\)

\(\text{ex.2: } ab = 0 \sim a = 0 \lor b = 0\) \(Q: \text{own}\)

\(\text{ex.3: } |a| = |b| \sim a = b \lor a = -b\) \(\text{111=111 true?}\)

\[\begin{array}{c|c|c}
T & \sim P & \text{\& and } \sim Q \text{ (also denoted } P \land \sim Q) \\
T & T & T \\
\end{array}\]
\[ \text{let } 3 \leq x \leq 4 \Leftrightarrow 3 \leq x \land x \leq 4 \]

\( x = 1 \)
\[ 3 \leq 1 \quad 1 \leq 4 \quad 3 \leq x \leq 4 \]
\[ \text{F} \quad \text{T} \quad \text{F} \]

\( x = \frac{3}{2} \)
\[ 3 \leq \frac{3}{2} \quad \frac{3}{2} \leq 4 \]
\[ \text{T} \quad \text{T} \quad \text{T} \]

\( \overline{\text{NOT}} \)
\[ \begin{array}{c|c}
P & \overline{P} \\
\hline
\text{T} & \text{F} \\
\text{F} & \text{T}
\end{array}
\] (also denoted \( \neg P \))

\( \text{let } P \rightarrow x > 3 \) then \( \neg P \rightarrow x \leq 3 \)

\( x = 2 \)
\[ 2 > 3 \quad \neg (2 > 3) \]
\[ \text{F} \quad \text{T} \]

Given a proposition you should be able to:
1. Reduce it to a combination of $\lor$, $\land$, $\neg$.

2. Decide if it's true or false using the truth table.

\[ P \land (Q \lor R) \]

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$Q \lor R$</th>
<th>$P \land (Q \lor R)$</th>
</tr>
</thead>
<tbody>
<tr>
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$3 \leq 4 \land (\pi \geq 3 \lor 5 = 6)$ is $\mathbf{T}$.
Lecture 2, 09/30/2019

Recap: Are \( \neg (p \lor q) \) and \( \neg p \land \neg q \) the same prop.?

Q: Table of truth:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg p )</th>
<th>( \neg q )</th>
<th>( p \lor q )</th>
<th>( \neg (p \lor q) )</th>
<th>( \neg p \land \neg q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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<td>F</td>
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Some table of truth

Ex: \( \neg (2n \geq 3 \lor s > b) \) same as \( (2n < 3) \land (s \leq b) \)

Put a square box at the end of a proof!

Chapter 2: Implications

Suppose \( P(n): n \geq 3 \) and \( Q(n): n > 0 \)
Let's see if this makes sense with our experience by checking for every \( m \):

<table>
<thead>
<tr>
<th>( m )</th>
<th>( P(m) )</th>
<th>( A(m) )</th>
<th>( P \rightarrow A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m \leq 0 )</td>
<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
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<tr>
<td>( 0 &lt; m \leq 3 )</td>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
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<tr>
<td>( m &gt; 3 )</td>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
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In words: There is no value of \( m \) for which \( P(m) \) is true and \( A(m) \) is false.

Indeed: \( P \rightarrow A \) comes as \( \neg (P \land \neg A) \).
\textbf{Proof}:

<table>
<thead>
<tr>
<th></th>
<th>( P )</th>
<th>( Q )</th>
<th>( \neg Q )</th>
<th>( P \wedge (\neg Q) )</th>
<th>( \neg (P \wedge (\neg Q)) )</th>
<th>( P \Rightarrow Q )</th>
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\textbf{Notation}:

\( P \Rightarrow Q \) :  
- \( P \) implies \( Q \),  
- \( Q \) if \( P \),  
- \( P \) only if \( Q \),  
- \( P \) is sufficient for \( Q \),  
- \( Q \) is necessary for \( P \).

\textbf{Equivalence}:

\( P \iff Q \) means \( P \Rightarrow Q \wedge Q \Rightarrow P \)

\textbf{Notation}:

\( P \iff Q \) :  
- \( P \) if and only if \( Q \)  
- \( P \) is equivalent to \( Q \)  
- \( P \) is necessary and sufficient for \( Q \)
We want to study the main ways one can prove that $P \Rightarrow Q$ is true.
Direct proof: We suppose that \( P \) is true and we show that \( Q \) is true.

\[
\begin{array}{c|c|c}
P & Q & P \rightarrow Q \\
\hline
\top & \top & \top \\
\end{array}
\]

Examples:

1. \( a = 1 \lor a = 2 \implies a^2 - 3a + 2 = 0 \)

Proof: We are going to use \((P \vee a \implies R) \iff (P \implies R) \land (a \implies R)\)

\[
\begin{align*}
a = 1 & \implies a^2 - 3a + 2 = 1 - 3 + 2 = 0 \\
a = 2 & \implies a^2 - 3a + 2 = 4 - 6 + 2 = 0
\end{align*}
\]

2. For positive real numbers \( a \) and \( b \),

\[
a < b \implies a^2 < b^2
\]

Proof: You have to do a bit of translating.

P: \( a, b > 0 \) real numbers

\[
\begin{array}{c}
\land \\
a < b
\end{array} \implies Q: a^2 < b^2
\]
\[ p \Rightarrow a^2 < ab \land ab < b^2 \]
\[ R \]
\[ R \Rightarrow a^2 < b^2 \]

(3) In words: Assume \( a, b > 0 \) and \( a < b \).

Multiply \( a < b \) by \( a, b \) respectively and using \( a, b > 0 \) we get \( a^2 < ab \) and \( ab < b^2 \). Chainning these two inequalities the conclusion follows.

(3) There is yet another wording on the back.

Remark: We used \([[(P \Rightarrow R) \land (R \Rightarrow Q)] \Rightarrow P \Rightarrow Q]\).
\[ \text{Proof: } a \text{ is a real number } \land a \neq 0 \implies a^2 > 0 \]

\[ a \neq 0 \implies a > 0 \lor a < 0. \]

\[ a > 0 \implies a^2 > 0 \]
\[ a < 0 \implies a^2 < 0 \quad (\forall v \in R) \implies S \]

\[ \text{We use } \left\{ \left[ P \implies (\forall V \in R) \right] \land \left[ (a \Rightarrow S) \land (\neg \Rightarrow S) \right] \right\} \]
\[ \implies (P \Rightarrow S) \]

\[ \text{(ii) For real numbers } a, b, \quad a < b \implies a b < (a + b)^2 \]

\[ \text{(iii) Direct proof backward: } \]
\[ ab < (a + b)^2 \implies ab < a^2 + 2ab + b^2 \]
\[ \implies a^2 + b^2 - 2ab > 0 \]
\[ \implies (a - b)^2 > 0 \]
\[ \implies a \neq b \]
\[ \implies a < b \]
Proof by contradiction

Want to prove \( P \) is true

Assume \( \neg P \) and prove a false proposition \( Q \)

\[ \neg P \Rightarrow Q \]

\[
\begin{array}{cccc}
  P & \neg P & Q & P \Rightarrow Q \\
  T & F & F & T \\
  F & T & T & T \\
\end{array}
\]

So we know \( Q \) is false

\[ \neg P \Rightarrow Q \] is true

\[
\begin{array}{cccc}
  P & \neg P & Q & P \Rightarrow Q \\
  T & F & F & F \\
  F & T & T & T \\
\end{array}
\]

Therefore we are in row 2, \( \neg P \Rightarrow Q \) is true.

\[ \therefore P \] is true

Example: 101 is an odd integer

First some definitions: