# Quiver representation

A quiver $Q$ is a directed graph.

$Q_0$ : set of vertices

$Q_1$ : set of arrows

Rep$(Q)$: category of finite-dim of $Q$

path algebra : $kQ$

Eg. $Q = 1 \xrightarrow{\alpha} 2$

Representations of $Q$ over $\mathbb{C}$ : $\mathbb{C}^m \xrightarrow{f} \mathbb{C}^n$

Consider (right) modules over path alg. $\mathbb{C}Q^{op}$

$$M \circlearrowleft \mathbb{C}Q$$
$$n \cdot a$$

$V = V_1 \xrightarrow{V_\alpha} V_2$ rep. of $Q$

$M = V_1 \oplus V_2$ as vector space

$$(v_1, v_2)$$
$$(v_1, v_2) e_1 = (v_1, 0)$$
$$(v_1, v_2) e_2 = (0, v_2)$$
$$(v_1, v_2)\alpha = (0, V_\alpha(v_1))$$

This defines functors

$$\boxed{\mathrm{rep}Q \rightleftarrows \mathrm{mod}\,\mathbb{C}Q^{op}}$$

Conversely, if $M$ is a module

$Me_1 \xrightarrow{M\alpha} Me_2$ $\quad e_1\alpha = \alpha$, $e_2\alpha = 0$

$\hookrightarrow$ inj. / proj. module $\to$ rep.

let $A$ be a finite-dim $\mathbb{C}$-alg

let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents

i.e.
- $e_1 + e_2 + \cdots + e_n = 1_A$
- cannot write $e_i$ as sum of idempotents non-trivially
- $e_i \cdot e_j = 0$ if $i \neq j$
- $e_i \cdot e_i = e_i$

Then $e_i A$ is a (right) proj. indecomposable modules

$$e_1 A \oplus \cdots \oplus e_n A = A$$

$Q: 1 \leftarrow 2 \leftarrow 3$

Claim:
$$\begin{cases} P_3: \mathbb{C} \leftarrow \mathbb{C} \leftarrow \mathbb{C} \\ P_2: \mathbb{C} \leftarrow \mathbb{C} \leftarrow 0 \\ P_1: \mathbb{C} \leftarrow 0 \leftarrow 0 \end{cases}$$

$Q^{op}: 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$

$P_1 = Ae_1 = \mathrm{Span}(e_1, e_2, e_3, \alpha, \beta, \beta\alpha) \cdot e_1$

$= \mathrm{Span}(e_1)$

$$\begin{array}{ccccc} P_1 \cdot e_1 & \leftarrow & P_1 \cdot e_2 & \leftarrow & P_1 \cdot e_3 \\ \| & & \| & & \| \\ \mathbb{C} & & 0 & & 0 \end{array}$$

$P_2 = A \cdot e_2 = \mathrm{Span}(e_2, \alpha)$

$$\begin{array}{ccccc} P_2 \cdot e_1 & \leftarrow & P_2 \cdot e_2 & \leftarrow & P_2 \cdot e_3 \\ \| & & \| & & \| \\ \mathbb{C} & & \mathbb{C} & & 0 \end{array}$$

$P_3 = Ae_3 = \mathrm{Span}(e_3, \beta, \beta\alpha)$

$$\begin{array}{ccccc} P_3 \cdot e_1 & \leftarrow & P_3 e_2 & \leftarrow & P_3 e_3 \\ \| & & \| & & \| \\ \mathbb{C} & & \mathbb{C} & & \mathbb{C} \end{array}$$

## Projective resolution

Let $A$ be a finite-dim $\mathbb{C}$-alg (e.g. $\mathbb{C}Q^{op}$)
$M$ finite-dim (right)-module over $A$.
( A free module is a moduli isom to $\bigoplus_{i\in I} A$)

**Prop** There exists an integer $m$ and a surjection
$$A^m \twoheadrightarrow M \to 0$$

**Prop** $P$ is proj. $\Leftrightarrow$ $P$ is a direct summand of a free module
$P$ is proj. indec $\Leftrightarrow$ $P$ . . . . of $A$

e.g. $A = e_1 A \oplus \cdots \oplus e_n A$
$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0 \quad \text{proj. resol.}$$

e.g. $M = 0 \leftarrow \mathbb{C} \xleftarrow{1} \mathbb{C}$

$$\begin{tikzcd}
0 \arrow[d] & & & \\
P_1 \arrow[d] \arrow[r, phantom, "="] & \mathbb{C} \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
P_3 \arrow[d] \arrow[r, phantom, "="] & \mathbb{C} \arrow[d] & \mathbb{C} \arrow[l, "1"'] \arrow[d] & \mathbb{C} \arrow[l, "1"'] \arrow[d] \\
M \arrow[d] \arrow[r, phantom, "="] & 0 & \mathbb{C} \arrow[l] & \mathbb{C} \arrow[l] \\
0 & & &
\end{tikzcd}$$

## Space of representation

e.g. $Q: 1 \leftarrow 2$. Fix a dim vector $\underline{d} = (2,3)$
A repn w/. dim vector $\underline{d}$ is
$$\mathbb{C}^2 \xleftarrow{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}} \mathbb{C}^3$$

so the variety of repn of $Q$ w/. dim vector $\underline{d}$ is $\subseteq \mathbb{A}^6$

ex. $\underline{d}=(1,1)$ $\quad M: \mathbb{C} \xleftarrow{\lambda} \mathbb{C}$

if $\lambda=0$, then $M = \begin{matrix} \mathbb{C} \xleftarrow{0} 0 \\ \oplus \\ 0 \xleftarrow{0} \mathbb{C} \end{matrix}$

$\lambda \neq 0$, then $M \cong \mathbb{C} \xleftarrow{1} \mathbb{C}$

isom. class

$$\begin{tikzcd} M = \mathbb{C} \arrow[d, "\lambda^{-1}"'] & \mathbb{C} \arrow[l, "\lambda\neq 0"'] \arrow[d, "1"] \\ \mathbb{C} & \mathbb{C} \arrow[l, "1"] \end{tikzcd}$$

$\text{rep}_{(1,1)}(Q) = \mathbb{A}^1$ $\quad \underset{0}{\sim\sim\sim} \ \lambda$

$\uparrow$ generic.

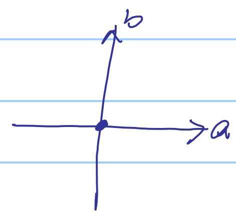
$\underline{d}=(2,1)$ $\quad M = \mathbb{C}^2 \xleftarrow{\binom{a}{b}} \mathbb{C}$ $\quad \text{rep}_{\underline{d}}(Q) \cong \mathbb{A}^2$

$a=b=0$ $\quad M = \begin{matrix} (\mathbb{C} \leftarrow 0)^2 \\ \oplus \\ (0 \leftarrow \mathbb{C}) \end{matrix}$

$a \neq 0, b=0$ $\quad M \cong \begin{matrix} \mathbb{C} \xleftarrow{a} \mathbb{C} \\ \oplus \\ \mathbb{C} \leftarrow 0 \end{matrix}$

$a=0, b\neq 0$ $\quad M \cong \begin{matrix} \mathbb{C} \leftarrow 0 \\ \oplus \\ \mathbb{C} \xleftarrow{b} \mathbb{C} \end{matrix}$

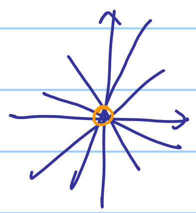
$a, b \neq 0$ $\quad M \cong \begin{matrix} \mathbb{C} \xleftarrow{1} \mathbb{C} \\ \oplus \\ \mathbb{C} \leftarrow 0 \end{matrix}$




e.g. $1 \leftleftarrows 2$ $\quad \underline{d}=(1,1)$

$\mathbb{C} \underset{\mu}{\overset{\lambda}{\leftleftarrows}} \mathbb{C}$

$\text{rep}_{\underline{d}}(Q) \cong \mathbb{A}^2$



No generic class

- $(0,0)$ is an isom. class
- Any line is an isom class

# generic proj. presentation

e.g. $Q = 1 \leftarrow 2 \leftarrow 3$

Look at morphism $P_1 \to P_3$

$$\begin{tikzcd}
P_1 \arrow[d, "f"'] & = & \mathbb{C} \arrow[d, "\lambda"'] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
P_3 \arrow[d] & = & \mathbb{C} & \mathbb{C} \arrow[l] & \mathbb{C} \arrow[l] \\
\operatorname{coker} f & & & &
\end{tikzcd}$$

$\mathrm{Hom}(P_1, P_3) \cong \mathbb{C} \cong \mathbb{A}^1$

The generic coker $(\lambda \neq 0)$ is $\cong 0 \leftarrow \mathbb{C} \xleftarrow{1} \mathbb{C}$

e.g. $Q : 1 \leftleftarrows 2$

$P_1 : e_1 \mathbb{C}Q^{op} = \mathbb{C} \leftleftarrows 0$

$P_2 : e_2 \cdot \mathbb{C}Q^{op} = \mathbb{C}^2 \overset{\binom{1}{0}}{\underset{\binom{0}{1}}{\leftleftarrows}} \mathbb{C}$ $\qquad \lambda \neq 0$ or $\mu \neq 0$

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r] \arrow[dr] & P_2 \arrow[r] & \left(\mathbb{C} \overset{\lambda}{\underset{\mu}{\leftleftarrows}} \mathbb{C}\right) \arrow[r] & 0 \\
& & \mathbb{C} \leftleftarrows 0 \arrow[u] & &
\end{tikzcd}$$

so if $Q$ is a quiver, $\underline{d}$ a dim vector

then there is a "generic proj. presentation" for modules in $\mathrm{rep}_{\underline{d}}(Q)$