Introduction to cluster algebra

- A cluster alg. is a certain alg. of $k(x_1,\dots,x_n)$
- generators constructed inductively via process called mutation.

E.g. $z_n = \frac{z_{n-1}+1}{z_{n-2}}$

$x, y, \frac{y+1}{x}, \quad \frac{\frac{y+1}{x}+1}{y} = \frac{x+y+1}{xy}$

$\frac{\frac{x+y+1}{xy}+1}{\frac{y+1}{x}} = \frac{x+y}{y}$

$\frac{\frac{x+y}{y}+1}{\frac{x+y+1}{xy}} = x, y, \dots$

Def. An $n\times n$ integer $B=(b_{ij})$ is skew-symmetrizable if $\exists\, d_1,\dots,d_n \in \mathbb{Z}^+$ s.t. $d_i b_{ij} = -d_j b_{ji} \ \forall i,j$

$\rightsquigarrow$ cluster alg. $\mathcal{A}(B)$ to B.
assoc.

Start with seed $\Sigma = (\underbrace{\{x_1,\dots,x_n\}}_{\text{cluster}}, B)$ ← cluster var. ; ← exchange matrix

$\Sigma \xrightarrow{\text{mutation}}$ get more seed

Column of B encode exchange relation

For $k=1,\dots,n$, $\quad x_k x_k' = \prod_{b_{ik}>0} x_i^{|b_{ik}|} + \prod_{b_{ik}<0} x_i^{|b_{ik}|}$

Define new var. x_k'

$M_k(\Sigma) = (\{x_1,\dots,\hat{x}_k,\dots,x_n\}\cup\{x_k'\}, \mu_k(B))$

where

$$\mu_k(B) = \begin{cases} -b_{ij} & k=i \text{ or } j \\ b_{ij} & b_{ik}\cdot b_{kj} \le 0 \\ b_{ij}+b_{ik}b_{kj} & b_{ik}, b_{kj} \ge 0 \\ b_{ij}-b_{ik}b_{kj} & b_{ik}, b_{kj} \le 0 \end{cases}$$

Rk: • $\mu_k(B)$ is skew-sym.
• $\mu_k \circ \mu_k = id.$

$$\begin{array}{ccc} \diagdown & & \diagup\ \mu_1(\Sigma) \diagdown \\ & \Sigma & \\ \cdots\diagup & & \diagdown\ \mu_2(\Sigma) \end{array}$$

exchange graph

Def. The cluster alg. $\mathcal{A}(B)$ is a subalg. of $k(x_1 \ldots x_n)$ generated by all cluster alg.

E.g. Rank 2 cluster alg.

let $\mathcal{F} = k(x_1, x_2)$

Fix $b, c \in \mathbb{Z}^+$. Define $B = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$

$$x_1 \underset{\substack{\| \\ x_3}}{x_1'} = x_2^c + 1$$

$$x_2 \underset{\substack{\| \\ x_0}}{x_2'} = x_1^b + 1$$

$$\mu_1(B) = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} = \mu_2(B)$$

$\mu_1^2 = \mu_2^2 = id.$ $\quad \mu_1\mu_2(B) = \mu_2\mu_1(B) = B$

$\therefore$ cluster variable is $\{x_m\}_{m \in \mathbb{Z}}$

$$x_{m-1}x_{m+1} = \begin{cases} x_m^b + 1 & x \text{ odd} \\ x_m^c + 1 & x \text{ even} \end{cases}$$

$\mathcal{A}(b,c) \subseteq \mathbb{Q}(x_1, x_2)$ generated by all x_i.

Actually x_i' are Laurant poly of x_1, x_2

$$x_i \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}] \qquad \forall i$$

Thm $(x_i)_{i\in\mathbb{Z}}$ in $\mathcal{A}(b,c)$ is periodic $\Leftrightarrow$ $bc \le 3$
correspond to cluster type A_2, B_2, G_2

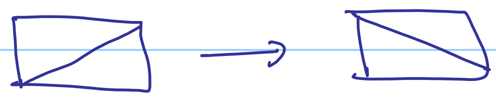
Fact 1: Laurent phenomenon.
given any seed for any cluster alg. & cluster var. x,
one can express x as Laurent poly in orig. var's

Fact 2: finite type cluster alg = only finitely many cluster variable
$\updownarrow$
Dynkin diagram

Triangulation

Consider a polygon with $(n+3)$-sides
Choose any triangulation T.

- all T are connected by flips

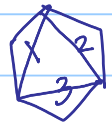


associate
$\rightsquigarrow$ $n\times n$ matrix $B(T)$ to T.

First label diag. of T by $1,\dots,n$

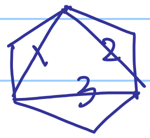
$B(T) = (b_{ij})$ $\quad (b_{ij}) = \#\{$triangles with sides i, j s.t.

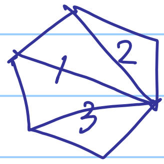
 clockwise$\}$

E.g. $T=$ 

$$B(T) = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

This gives cluster alg $\mathcal{A}(B(T))$ assoc. to each tri. of polygon.



$\xrightarrow{\mu_1}$


T'

$$\mu_1(B(T)) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Claim $\mu_i(B(T)) = B(T')$

$\uparrow$ by flipping i

Coro The cluster algebra $\mathcal{A}(B(T))$ does not depend on T, only on $\# = n+3$.

Thm We have bijection

cluster var. $\longleftrightarrow$ diagonal

cluster $\longleftrightarrow$ triangulation

Exchange rel. $\longleftrightarrow$ flips

Def (more general) let B be $m\times n$ integer matrix $(m \geq n)$

$\left(\boxed{} \right) \leftarrow$ $n\times n$ part is skew-sym

Start with initial seed.

$$\Sigma = \Big(\underbrace{\{ \underbrace{x_1 \ldots x_n}_{\text{cluster}}, \underbrace{x_{n+1}, \ldots x_m}_{\text{coeff cluster}} \}}_{\text{extended cluster.}}, B \Big)$$

For $k = 1, \dots, n$, get new var.

$$\left(\{ x_1, \dots, \hat{x}_k, \dots, x_n, x_k', x_{n+1}, \dots, x_m \}, \mu_k(B) \right)$$

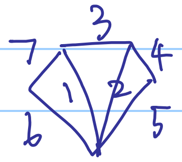
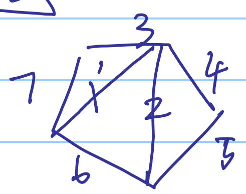
where
$$\mu_k(B) = \begin{cases} -b_{ij} & k = i \text{ or } j \\ b_{ij} & b_{ik} \cdot b_{kj} \le 0 \\ b_{ij} + b_{ik} b_{kj} & b_{ik}, b_{kj} \ge 0 \\ b_{ij} - b_{ik} b_{kj} & b_{ik}, b_{kj} \le 0 \end{cases}$$

and x_k' is defined by
$$x_k x_k' = \prod_{b_{ik} > 0} x_i^{|b_{ik}|} + \prod_{b_{ik} < 0} x_i^{|b_{ik}|}$$

Def. A cluster monomial is a monomial in the elt. of some extended cluster (in a fixed seed)

Triangulation of $(n+3)$-gon
label diag. as $1, \dots, n$
bdry as $n+1, \dots, m$
$\therefore m - n = n + 3$

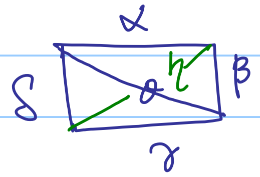
$B(T) = (b_{ij})$ $\quad b_{ij} = \#\{$triangles with sides i & j with j following i clockwise$\}$

$T =$  $\xrightarrow{\text{flip } 1}$ 

$n = 2$

$$B(T) = \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 0 & \\ -1 & 0 \\ 1 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \end{array} \longmapsto \mu_1(B(T)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & -1 \\ -1 & 0 \end{pmatrix}$$

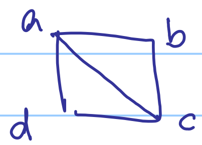
Rk: For each rectangle



$\theta\eta = \alpha\gamma + \beta\delta$

The cluster alg. with coeff. assoc. to $(n+3)$-gon can be identified with coord. ring of $Gr_{2,n+3}$.
Call this cluster A_n

Fact: Coord. ring of $\mathbb{C}[Gr_{2,n}]$ is generated by p_{ij} subject to the relation $\quad p_{ac}p_{bd} = p_{ab}p_{cd} + p_{ad}p_{bc}$



where p_{ij} are Plucker coord.

$p_{ij}(A)$ = det of 2×2 submatrix of A in col.'s i & j.

e.g. $A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$

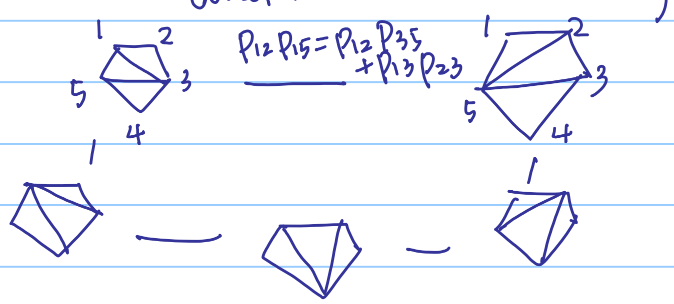
$p_{34}(A) = ad - bc$

Exchange graph of A_{n-3} is the graph whose vertices are triangulations connected by flips

e.g. $Gr_{2,5}(\mathbb{C})$

coord. ring is gen. by $p_{12}, p_{13}, \ldots p_{45}$

and relation encoded by



In general, the exchange graph of A_n is the 1-skeleton of the convex polytope called associahedron.

For A_n, the cluster monomials
$$= \{\text{monomials in Plücker coord. which is supported on tri}\}$$

Thm (old thm) The cluster monomials for $A_{n-3} \cong \mathbb{C}[Gr_{2,n}]$ form an additive basis for A_{n-3}.