Quiver repn & cluster algebra

Q: finite quiver
Q_0: vertices $\quad Q_1$: arrows
k: alg. closed field

V is simple $\Leftrightarrow$ $V \neq 0$ and each subrep equals to 0 or V.
V is indecomposable $\Leftrightarrow$ $V \neq 0$ and $\forall U, W$ s.t. $V \cong U \oplus W \Rightarrow U$ or $W = 0$

Example with infinitely indec. rep./isom.

$$1 \rightrightarrows 2$$

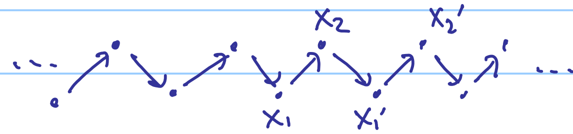
Then $V(x) = \left(k \underset{x_1}{\overset{x_0}{\rightrightarrows}} k \right) \qquad (x_0 : x_1) \in \mathbb{P}^1(k)$

$\therefore$ inf family of pairwise non-isom rep.

Thm Q has only finitely many indec. up to isom.
$\Leftrightarrow$ Q is a Dynkin quiver.
if $Q = \vec{\Delta}$, then
$\{\text{indec. rep. of } Q\}/\text{isom} \cong \{\text{pos. root. of } \Delta\}$
$$V \mapsto \sum_{i=1}^{n} (\dim V_i)\, \alpha_i$$
α_i : simple roots

Knitting algorithm

$\Delta = A_2 = \circ\!\!-\!\!\!-\!\!\circ$
make it $\vec{\Delta} = A_2 : 1 \to 2$
Repetition $= \mathbb{Z}\vec{\Delta}$
First consider $\mathbb{Z} \times \vec{\Delta}$
Then: for $\alpha : i \to j$
add new family of arrows
$$(n, \alpha^*) : (n, j) \to (n+1, i) \qquad \forall\, n \in \mathbb{Z}$$



Assign a cluster variable to each vertex of the repetition

at zero-th copy: x_1, x_2

$$x_1' = \frac{1+x_2}{x_1} \quad \longleftrightarrow \quad \alpha_1$$

$$x_2' = \frac{1+x_1'}{x_2} = \frac{1+x_1x_2}{x_1x_2} \longleftrightarrow \alpha_1+\alpha_2$$

$$x_1'' = \frac{1+x_1}{x_2} \quad \longleftrightarrow \quad \alpha_2$$

$$x_2'' = x_1, \quad x_1''' = x_2$$

$\therefore$ 5 cluster var.

$\mathcal{A}_{A_2}$ is a $\mathbb{Q}$-subalg. of $\mathbb{Q}(x_1, x_2)$ gen. by 5 variables.

- The computation is periodic $\Rightarrow$ finitely many cluster var.
- $5 = 2 + 3$ ← non initial var. x_1', x_2', x_1''

  2 ↑ initial var.

  $\updownarrow$ bij

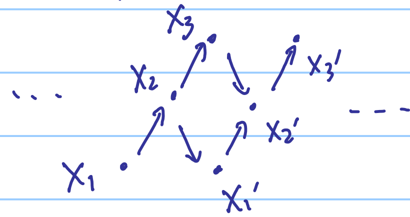
  pos. root in root system $\Delta = \{\alpha_1, \alpha_2\}$

  look at denominators

  $x_1^{d_1} x_2^{d_2}$ corr. $d_1\alpha_1 + d_2\alpha_2$

A_3

$$1 \longrightarrow 2 \longrightarrow 3$$



$$x_1' = \frac{1+x_2}{x_1}$$

$$x_2' = \frac{1+x_1'x_3}{x_2} \quad \cdots$$

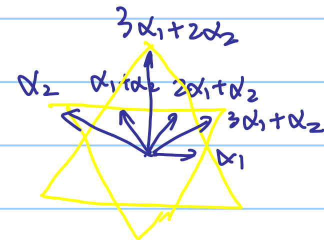
9 variables
$\shortparallel$
$3+6$

- each slide gives a seed, but NOT all seed from slide (e.g. x_1, x_3, x_1'')

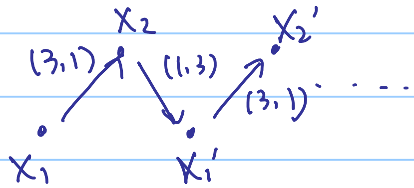
G_2

(3,1)

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$



$$\vec{\Delta} : 1 \xrightarrow{(3,1)} 2$$



$$x_1' = \frac{1+x_2}{x_1}$$
$$x_2' = \frac{1+(x_1')^3}{x_2}$$
$$\vdots$$

$$8 = 2+6$$

Mutation

Seed (R, u)

R : finite quiver without loop or 2-cycles with vertex set $\{1, \dots, n\}$
u : free generating set $\{u_1, \dots, u_n\}$ of the field $\mathbb{Q}(x_1, \dots, x_n)$
of fraction of the poly ring $\mathbb{Q}[x_1, \dots, x_n]$ in n indeterminates

Fix a vertex k
mutation at $k = \mu_k(R, u) = (R', u')$
a) R' get from R:
1) reverse all arrow incident with k
2) $\forall$ vertex $i \neq j$, distinct from k, modify the number of arrows between i & j as

R	R'
$i \xrightarrow{r} j$, $i \xrightarrow{s} k$, $k \xrightarrow{t} j$	$i \xrightarrow{r+st} j$, $k \xrightarrow{s} i$, $j \xrightarrow{t} k$
$i \xrightarrow{r} j$, $k \xrightarrow{s} i$, $j \xrightarrow{t} k$	$i \xrightarrow{r-st} j$, $i \xrightarrow{s} k$, $k \xrightarrow{t} j$

b) u' is obtained from u by replacing u_k with

$$u_k = \frac{1}{u_k}\left(\prod_{\substack{\text{arrow}\\ i\to k}} u_i + \prod_{\substack{\text{arrow}\\ k\to j}} u_j\right)$$

E.g.

$$R = \begin{tikzcd} & 1 \arrow[dr] & \\ 2 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd}$$

$$u = \{x_1, x_2, x_3\}$$

μ_1:

$$\begin{tikzcd} & 1 \arrow[dl] & \\ 2 \arrow[rr, dotted, no head, "1-1=0"'] & & 3 \arrow[ul] \end{tikzcd}$$

$$u_1' = \frac{x_2 + x_3}{x_1}, \quad u_2' = u_2 = x_2, \quad u_3' = u_3 = x_3$$

μ_1:

$$\begin{tikzcd} & 1 & \\ 2 \arrow[ur] & & 3 \arrow[ul] \end{tikzcd}$$

$$u_1'' = u_1' = \frac{x_2 + x_3}{x_1} \qquad u_2'' = \frac{x_1 + x_2 + x_3}{x_1 x_2}, \quad u_3'' = u_3' = x_3$$

Thm (Fomin-Zelevinsky)

Q: finite connected quiver without loops or 2 cycles with vertex set.

$\mathcal{A}_Q$: associated cluster algebra

\# cluster var. is finite

$\Leftrightarrow$ Q is mutation equiv. to an orientation of a simply laced Dynkin diagram Δ.

if $Q = \vec{\Delta}$

$$\{\text{positive root of } \Delta\} \leftrightarrow \{\text{non-initial cluster var.}\}$$

$$\underbrace{\alpha = \sum d_i \alpha_i}_{\alpha_i \text{ simple}} \quad \leftrightarrow \quad x_\alpha = \frac{\sim\sim}{x_1^{d_1} \cdots x_n^{d_n}}$$

- Knitting algorithm yields all cluster variables
  $\Leftrightarrow$ the quiver Q has two vertrices or is a orientation of a simply laced Dynkin diagram Δ.

Cor if $Q = \vec{\Delta}$,
then $\{\text{indecomp. rep.}\}/\text{isom} \xrightarrow{\simeq} \{\text{non initial cluster var.}\}$

$$V \mapsto x_V = \frac{\sim\sim}{x_1^{d_1} \cdots x_n^{d_n}}$$

$$d_i = \dim V_i \ \forall i$$

Cluster with coeff.

Def. ice quiver of type (m,n) is a quiver with vertex set

$$\underbrace{\{1,\dots,m\}}_{\tilde{Q}} = \underbrace{\{1,\dots,n\}}_{Q} \cup \underbrace{\{n+1,\dots,m\}}_{\text{forzen vertices.}}$$

quiver: $\tilde{Q}$ $\quad$ Q $\quad$ forzen vertices.

Cluster var: $\quad \underset{\text{cluster var.}}{u_1,\dots,u_n} \qquad \underset{\text{coeff.}}{x_{n+1},\dots,x_m}$

Cluster type of $\tilde{Q}$ is that of Q

E.g. $SL(2,\mathbb{C})$

$\sim \mathbb{C}[a,b,c,d]/(ad-bc-1)$

this alg. has a cluster alg. structure isom. to cpx of cluster alg. with coeff. assoc. to the following ice quiver

$$\begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ \boxed{2} & & \boxed{3} \end{tikzcd} \qquad A_1$$

$\Leftarrow$ forzenvertrices

Only 1 mutation:

$$\begin{tikzcd} & 1 & \\ 2 \arrow[ur] & & 3 \arrow[ul] \end{tikzcd}$$

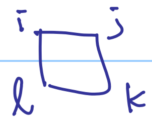
$$x_1 x_1' = 1 + x_2 x_3$$

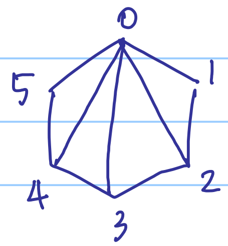
$$\underset{a}{x_1}\,\underset{d}{x_1'} - \underset{b}{x_2}\,\underset{c}{x_3} = 1$$

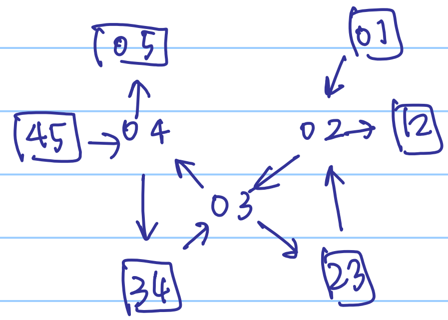
Note: this cluster structure is not unique.

E.g. $Gr_{2,n+3}(\mathbb{C})$

$$\sim \mathbb{C}[x_{ij}]_{1\le i<j\le n+3} \Big/ x_{ik}x_{jl} = x_{ij}x_{kl} + x_{jk}x_{il}$$







arrow = exchange relation appear when replace [ik] by flips

e.g. [03][24] = [04][23] + [02][34]