Quiver repn & cluster algebra

\( Q \): finite quiver
\( Q_0 \): vertices \( Q_1 \): arrows
\( \mathbb{K} \): alg. closed field

\( V \) is simple \( \iff V \cong \mathbb{K} \) and each subrep equals to 0 or \( V \).
\( V \) is indecomposable \( \iff V \cong \mathbb{K} \) and \( \forall U, W \text{ s.t. } V \cong U \oplus W \implies U \cong W = 0 \)

Example with infinitely indec. rep. / isom.

\[
\begin{array}{c}
1 \rightarrow 2 \\
\end{array}
\]

Then \( V(x) = \begin{pmatrix} \mathbb{K} & \mathbb{K} \\
\mathbb{K} & \mathbb{K} \end{pmatrix} \quad (x_0 : x_1) \in \text{FP}'(\mathbb{K}) \)

\( \vdots \) inf family of pairwise non-isom rep.

Thm. \( Q \) has only finitely many indec. up to isom.

\( \iff \) \( Q \) is a Dynkin quiver.

If \( Q = \Delta \), then

\( \{ \text{indec. rep. of } \Delta \}/\text{isom} \cong \{ \text{pos. root. of } \Delta \} \)

\( V \mapsto \sum \alpha_i (\dim V_i) x_i \)

\( \alpha_i \): simple roots

**Knitting algorithm**

\( \Delta = A_2 = \begin{array}{c} \end{array} \)

make it \( \overset{\Delta}{\Delta} = A_2 : 1 \rightarrow 2 \)

Repetition = \( \mathbb{Z} \Delta \)

First consider \( \mathbb{Z} \times \mathbb{Z} \)

Then for \( \alpha : i \rightarrow j \)

add new family of arrows

\( (n, \alpha^*) : (n, j) \rightarrow (n+1, i) \quad \forall n \in \mathbb{Z} \)
Assign a cluster variable to each vertex of the repetition at zero-th copy: \( x_1, x_2 \)

\[
\begin{align*}
x_1' &= \frac{1+x_2}{x_1} & \leftrightarrow & \quad x_1 \\
x_2' &= \frac{1+x_1'}{x_2} = \frac{1+x_1x_2}{x_1x_2} & \leftrightarrow & \quad x_1 + x_2 \\
x_1'' &= \frac{1+x_1}{x_2} & \leftrightarrow & \quad x_2
\end{align*}
\]

\[x_2'' = x_1, \quad x_1''' = x_2\]

\[\therefore \text{ 5 cluster var.}\]

\( A_{A_2} \) is a \( \mathbb{Q} \)-subalg of \( \mathbb{Q}(x_1, x_2) \) gen. by 5 variables.

- The computation is periodic \( \Rightarrow \) finitely many cluster var.

- 5 = 2 + 3
  
  \[\begin{array}{c}
  2 \\
  \downarrow \text{non initial var. } x_1', x_2', x_1''
  \end{array}\]
  
  \[\begin{array}{c}
  3 \\
  \downarrow \text{initial var.}
  \end{array}\]

- \( \text{pos. root in root system } \Delta = \{x_1, x_2\} \)

- Look at denominators
  
  \[x_1^{d_1} x_2^{d_2} \text{ corv. } d_1 x_1 + d_2 x_2\]

\( A_3 \)

\[
\begin{array}{c}
1 \rightarrow 2 \rightarrow 3 \\
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\vdots \quad x_i', \quad \vdots \quad x_i' \\
\end{array}
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\]

\[
\begin{array}{c}
\begin{array}{c}
\vdots \quad x_i, \quad \vdots \quad x_i \\
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\]

\[
\begin{align*}
x_1' &= \frac{1+x_2}{x_1} \\
x_2' &= \frac{1+x_1'x_3}{x_2} \ldots \\
&\quad \ldots \quad 9 \text{ variables}
\end{align*}
\]

- Each slide gives a seed, but not all seed from slide (e.g. \( x_1, x_3, x_i \))
\[ G_2 \]

\[ \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \]

\[ \Delta : 1 \rightarrow 2 \]

\[ x_2 \]
\[ x_2' \]

\[ (3,1) \]
\[ (1,1) \]
\[ (3,1) \]

\[ x_1 \]
\[ x_1' \]

\[ x_1' = \frac{1+x_2}{x_1} \]
\[ x_2' = \frac{1+(x_1')^3}{x_2} \]

\[ \vdots \]

\[ 8 = 2 + 6 \]

**Mutation**

**Seed** \((R, u)\)

- \(R\): finite quiver without loop or 2-cycles with vertex set \(\{1, \ldots, n\}\)
- \(u\): free generating set \(\{u_1, \ldots, u_n\}\) of the field \(\mathbb{Q}(x_1, \ldots, x_n)\)
- Fraction of the poly ring \(\mathbb{Q}[x_1, \ldots, x_n]\) in \(n\) indeterminates

Fix a vertex \(k\)

Mutation at \(k = \mu_k (R, u) = (R', u')\)

- \(R'\) get from \(R\).
  1) Reverse all arrow incident with \(k\)
  2) A vertex \(i \neq j\), distinct from \(k\), modify the number of arrows between \(i\) and \(j\) as
b) $w'$ is obtained from $w$ by replacing $uk$ with

$$u_k = \frac{i}{uk} \left( \prod_{i \rightarrow k} u_i + \prod_{k \rightarrow j} u_j \right)$$

E.g. \( R = \begin{array}{c}
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**Thm (Fomin-Zelevinsky)**

$\mathcal{Q}$: finite connected quiver without loops or 2 cycles with vertex set.

$A_{\mathcal{Q}}$: associated cluster algebra

- The number of cluster variables is finite
- $\mathcal{Q}$ is mutation equivalent to an orientation of a simply laced Dynkin diagram $\Delta$.

If $\mathcal{Q} = \Delta$

- \{positive root of $\Delta$\} $\leftrightarrow$ \{non-initial cluster variables\}
- $\alpha = \sum d_i \delta_i \leftrightarrow X_\alpha = \frac{n \ldots n}{x_1^{d_1} \ldots x_n^{d_n}}$
  - Simple

- Knitting algorithm yields all cluster variables
- $\mathcal{Q}$ has two vertices or is a orientation of a simply laced Dynkin diagram $\Delta$.

**Cor** If $\mathcal{Q} = \Delta$,

Then \{indecomp. rep.\}/isom $\leftrightarrow$ \{non-initial cluster variables\}

$V \rightarrow X_v = \frac{n \ldots n}{x_1^{d_1} \ldots x_n^{d_n}} \quad d_i = \dim V_i \forall i$
Cluster with coeff.

Definition: Ice quiver of type \((m,n)\) is a quiver with vertex set \(\{1, \ldots, m\} \cup \{1, \ldots, n\}\) such that:

- Quiver: \(\tilde{Q}\) with vertex set \(\{1, \ldots, m\} \cup \{n+1, \ldots, m\}\) and frozen vertices.
- Cluster var: \(u_1, \ldots, u_n, x_{n+1}, \ldots, x_m\).
- Cluster var. coeff.

Cluster type of \(\tilde{Q}\) is that of \(Q\).

Example: \(SL(2,\mathbb{C})\)

\[\sim \left[\begin{array}{cccc} a & b & c & d \\ \end{array}\right] / (ad-bc-1)\]

This alg. has a cluster alg. structure isom. to cpx of cluster alg. with coeff. assoc. to the following ice quiver:

\[\begin{array}{ccc} 1 & 2 & 3 \\ \end{array}\]

with frozen vertices:

Only 1 mutation:

\[x_1 x_{i-1} = 1 + x_2 x_3\]

\[x_1 x_i' - x_2 x_3 = 1\]

\[a \ b \ c \ d\]

Note: This cluster structure is not unique.
E.g. \( \text{Gr}_{2, n+3} (\mathbb{C}) \)

\[ \sim \mathbb{C} \langle \{X_{ij} \mid 1 \leq i, j \leq n+3 \} \rangle / X_{ik} X_{je} = X_{ij} X_{ke} + X_{jk} X_{le} \]

Arrow = exchange relation appear when replace \([i:k]\) by fups

E.g. \([03][24] = [04][23] + [02][34]\)