Categorification: Dynkin case

rep(Q): category of rep.

it is an abelian category: direct sum, kernels, cokernels are computed componentwise.

Def. A rep is simple if it is nonzero & only 2 subrep.

$\forall Q$ quiver, $\forall i$ vertex

define $(S_i)_j = \begin{cases} k & i=j \\ 0 & \text{else} \end{cases}$

A rep is indecomposable if $V = V' \oplus V'' \Rightarrow V'$ or $V'' = 0$

Thm All repn decomposes into finite sum of indecomp.

Thm finite number of indecom $\Leftrightarrow$ Q is simply laced Dynkin diagram Δ

Caldero-Chapoton formula

Δ: simply laced Dynkin diagram

$Q: \vec{\Delta}$

$$\left\{ \begin{array}{c} \text{indecomp. rep} \\ V = \bigoplus_{i \in Q_0} V_i^{d_i} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{non-initial cluster var.} \\ \frac{\cdots}{x_1^{d_1} \cdots x_n^{d_n}} \end{array} \right\}$$

e.g. $Q = \vec{A_2}$

$S_2 = (0 \to k)$ $\quad$ $P_1 = (k \to k)$ $\quad$ $S_1 = (k \to 0)$

$X_{S_2} = \dfrac{1+x_1}{x_2}$ $\quad$ $X_{P_1} = \dfrac{x_1+1+x_2}{x_1 x_2}$ $\quad$ $X_{S_1} = \dfrac{1+x_2}{x_1}$

Let Q be a finite quiver with vertices $1, \dots, n$

V: finite dim rep. of Q

d = dim vector of Q

$$CC(V) = \frac{1}{x_1^{d_1} \cdots x_n^{d_n}} \left(\sum_{0 \le e \le d} \chi(Gr_e(V)) \cdot \prod_{i=1}^{n} x_i^{\sum_{j \to i} e_j + \sum_{i \to j} (d_j - e_j)} \right)$$

$0 \le e \le d$ means $0 \le e_i \le d_i \ \forall i$

where $Gr_e(V)$ = variety of n-tuple of subspaces $U_i \subseteq V_i$ s.t. $\dim U_i = e_i$ & U_i form subrep of V

Derived category

$k = \bar{k}$

Q: Dynkin quiver

Q_0: set of vertices

kQ: $\bigoplus_{i,j \in Q_0} \underbrace{\mathrm{Hom}(i,j)}_{\text{path from } i \text{ to } j}$

$1 = \sum_{i \in Q_0} e_i \Rightarrow P_i = e_i kQ$

e_i: lazy path

$\{P_i\} \rightsquigarrow$ Category of k-finite dim right module $\underbrace{\text{mod}\, kQ}_{\text{abelian}}$

$$\mathrm{rep}_k(Q^{op}) \xrightarrow{\simeq} \text{mod}\, kQ$$

$\mathcal{D}_Q$: bounded derived category $D^b(\text{mod}\, kQ)$ of mod kQ

i.e. obj are $\cdots 0 \to \cdots M^p \xrightarrow{d^p} M^{p+1} \to$

$$\text{mod}\, kQ \hookrightarrow \mathcal{D}_Q$$

$$M \mapsto \quad \cdots \to 0 \to \underset{\deg 0}{M} \to 0 \to \cdots$$

If L and M are modules,

$$\mathrm{Ext}^i_{kQ}(L,M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_Q}(L, M[i]) \quad \forall i \in \mathbb{Z}$$

$\mathcal{D}_Q$: triangulated

Now consider $\mathbb{Z}Q$: repetition

For $\alpha : i \to j$, $\quad \sigma(p,\alpha) : (p-1, j) \to (p, i)$

For v vertex of $\mathbb{Z}Q$, the mesh ending at v is the full subquiver:

$$\begin{tikzcd}[row sep=small] & u_1 \arrow[dr, "\alpha"] & \\ \tau v \arrow[ur, "\sigma(\alpha)"] \arrow[dr] & \vdots & v \\ & u_s \arrow[ur] & \end{tikzcd}$$

Thm a) $\{\text{set of vertices of } \mathbb{Z}Q\} \to \{\text{set of isom. of indec. } \mathcal{D}_Q\}$

$$(1, i) \mapsto P_i$$

$$\rightsquigarrow \quad v \mapsto M_v$$

b) let $\mathrm{ind}\,\mathcal{D}_Q$ = full subcategory of indecomp. of $\mathcal{D}_Q$

(a) $\Rightarrow$ mesh category of $\mathbb{Z}Q \xrightarrow[\text{equiv.}]{\simeq} \mathrm{ind}\,\mathcal{D}_Q$

have triangle:

$$M_{\tau v} \to \bigoplus_{i=1}^{s} M_{u_i} \to M_v \to \Sigma M_{\tau v}$$

Auslander-Reiten triangle (almost split triangle)

if $M_v, M_{\tau v}$ are modules $\Rightarrow \bigoplus M_{u_i}$ is module too!

$$0 \to M_{\tau v} \to \bigoplus_{i=1}^{s} M_{u_i} \to M_v \to 0$$

AR seq. (almost split seq.)

$K_0(\mathcal{T})$ = Grothendieck group

$= \langle [X] \in \mathcal{T} \rangle / [X]-[Y]+[Z]$ from triangle

$\therefore\ K_0(\mathrm{mod}\, kQ) \xrightarrow{\sim} K_0(\mathcal{D}_Q)$

Since $K_0(\mathrm{mod}\, kQ)$ is free on $[S_i]$ S_i: simple rep

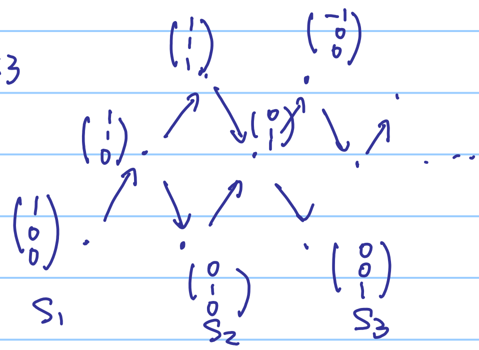
$\hookrightarrow$ same for $K_0(\mathcal{D}_Q)$

$\therefore$ elt given by $\mathbb{Z}^n$

denote $\underline{\dim} M$ dim vector of M

$$\underline{\dim} M_v = \sum_{i=1}^{s} \underline{\dim} M_{u_i} - \underline{\dim} M_{\tau v}$$

e.g. A_3



& τ yields a k-linear auto in $\mathcal{D}_Q$

$\tau\Sigma \xrightarrow{\sim} S$ S: Serre functor of $\mathcal{D}_Q$

Cluster category

$$\mathcal{C}_Q = \mathcal{D}_Q / (\tau^{-1}\Sigma)^{\mathbb{Z}} = \mathcal{D}_Q / (S^{-1}\Sigma^2)^{\mathbb{Z}}$$

morphism: $\mathcal{C}_Q(X,Y) = \bigoplus_{p\in\mathbb{Z}} \mathcal{D}_Q(X, (S^{-1}\Sigma^2)^p Y)$

orbit category of the derived category under $\tau^{-1}\Sigma$

def $\Rightarrow S \xrightarrow{\sim} \Sigma^2$

$\therefore$ $\mathcal{C}_Q$ is 2-CY

Thm $\left\{\begin{array}{l}\text{isom class}\\ \text{of indecomp of } \mathcal{C}_Q\end{array}\right\} \longrightarrow \{\text{cluster var. of } \mathcal{C}_Q\}$

$\Sigma P_i \longmapsto x_i$
shift proj. initial var.

$L \longmapsto x_L$

Eg: $Q: 1 \to 2$

$\mathrm{ind}(\mathrm{mod}\, kQ)$

$$\begin{tikzcd}[column sep=small] & P_2 \arrow[dr, two heads] & \\ P_1 \arrow[ur, hook] & & S_2 \end{tikzcd}$$

$\downarrow$

$\mathrm{ind}(\mathcal{D}_Q)$

$$\begin{tikzcd}[column sep=tiny] \cdots \arrow[dr] & & \tau P_2 \arrow[dr] & & P_2 \arrow[dr, two heads] & & \Sigma P_1 \arrow[dr] & & \Sigma S_2 \\ & \tau P_1 \arrow[ur] & & P_1 \arrow[ur, hook] & & S_2 \arrow[ur] & & \Sigma P_2 \arrow[ur] & \end{tikzcd}$$

$\downarrow$

$\mathrm{ind}(\mathcal{C}_Q)$

$$\begin{tikzcd}[column sep=small] & P_1 \arrow[r] & P_2 \arrow[dr] & \\ \Sigma P_2 \arrow[ur] & & & S_2 \arrow[dl] \\ & & \Sigma P_1 \arrow[ull] & \end{tikzcd}$$

$\downarrow$

cluster variable

$$\begin{tikzcd}[column sep=small, arrows=dash] & \frac{1+x_2}{x_1} \arrow[r] & \frac{1+x_1+x_2}{x_1x_2} \arrow[dr] & \\ x_2 \arrow[ur] & & & \frac{1+x_1}{x_2} \\ & x_1 \arrow[ul] \arrow[urr] & & \end{tikzcd}$$