Cluster categories (continue...)

Q: Dykin quiver with vertices $1, \dots, n$
V: finite dim rep. of Q
d = dimension vector

$\{\text{indecomp. rep } V = \oplus V_i^{d_i}\} \longrightarrow \{\text{non-initial cluster variable}\}$

$$V \mapsto \frac{1}{x_1^{d_1} \cdots x_n^{d_n}} \left(\sum_{\substack{0 \le e \le d \\ \uparrow \\ 0 \le e_i \le d_i \forall i}} \chi(Gr_e(V)) \prod_{i=1}^{n} x_i^{\sum_{j \to i} e_j + \sum_{i \to j} (d_j - e_j)} \right)$$

where $Gr_e(V)$ = variety of n-tuple of subspaces $U_i \subseteq V_i$
s.t. $\dim U_i = V_i$ & U_i form subrep. of V

E.g. $\bullet \longrightarrow \bullet$

dim vector $= (2,1)$
Fix the rep to be the decomp. rep which is a surj. map.
$V = \mathbb{C}^2 \to \mathbb{C}$

Submodule:	$(0,0)$,	$(1,0)$,	$(0,1)$,	$(1,1)$,	$(2,1)$
	x_1	x_1x_2	1	x_2	x_2^2

But for $(1,1)$, we have a choice of $\mathbb{P}^1$ family of the line in $\mathbb{C}^2$

$\therefore \quad X_V = \frac{1}{x_1^2 x_2}(1 + x_1 + 2x_2 + x_2^2 + x_1x_2)$

OR

$$\mathbb{C}^2 \to \mathbb{C} = \begin{pmatrix} \mathbb{C} \to \mathbb{C} \\ \oplus \\ \mathbb{C} \to 0 \end{pmatrix} \begin{matrix} \leftarrow \frac{1+x_1+x_2}{x_1x_2} \\ \\ \leftarrow \frac{1+x_2}{x_1} \end{matrix}$$

$$\left(\frac{1+x_1+x_2}{x_1x_2}\right)\left(\frac{1+x_2}{x_1}\right) = \frac{1+x_1+2x_2+x_1x_2+x_2^2}{x_1^2x_2} = X_V$$

E.g. 2 D_4

$$\begin{matrix} 3 \to & \\ 2 \to & 4 \\ 1 \nearrow & \end{matrix}$$

$$d = \left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\ 2\right)$$

Submodule:

$$\left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\ 0\right),\ \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\ 1\right)\times 3,\ \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\ 2\right)\times 3,\ \left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\ 2\right),\ \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\ 2\right)\times 3,\ \left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\ 1\right),\ \left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\ 2\right)$$

$$x_1^2x_2^2x_3^2 \quad x_1x_2x_3x_4 \quad x_4 \quad 1 \quad x_4^2 \quad x_1x_2x_3 \quad x_4^3$$

$\uparrow$ again $\mathbb{P}^1$

$$CC(V) = \frac{1}{x_1x_2x_3x_4^2}\left(1 + 3x_4 + 3x_4^2 + x_4^3 + 2x_1x_2x_3 + 3x_1x_2x_3x_4 + x_1^2x_2^2x_3^2\right)$$

Aim: Construct category $\mathcal{C}_Q$ whose indec. rigid objects are in bijection with all cluster variable of $\mathcal{A}_Q$.

$k = \mathbb{C}$

Q: Dynkin quiver

kQ: path algebra

$\mathrm{mod}(kQ) = \{k\text{-finite dim right } kQ\text{-modules}\}$

$\downarrow$

$\mathcal{D}_Q$ = bounded derived category

$\downarrow$

$\mathcal{C}_Q$: cluster cat. $= \mathcal{D}_Q / (\tau^{-1}\Sigma)^{\mathbb{Z}} \cong \mathcal{D}_Q / (S^{-1}\Sigma^2)^{\mathbb{Z}}$

$Q: 1 \to 2$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & P_2 \arrow[dr] & \\
P_1 \arrow[ur] & & S_2
\end{tikzcd}$$

$\mathrm{ind}(\mathrm{mod}\, kQ)$

$\hookdownarrow$

$\mathrm{ind}(\mathcal{D}_Q)$

$$\begin{tikzcd}[column sep=small, row sep=small]
\vdots & \tau P_2 \arrow[dr] & & P_2 \arrow[dr] & & \Sigma P_1 \arrow[dr] & & \Sigma S_2 & \cdots \\
\tau P_1 \arrow[ur] & & P_1 \arrow[ur] & & S_2 \arrow[ur] & & \Sigma P_2 \arrow[ur] & &
\end{tikzcd}$$

$\downarrow$

$\mathrm{ind}(\mathcal{C}_Q)$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & P_2 \arrow[dr] & \\
P_1 \arrow[ur] & & S_2 \arrow[d] \\
\Sigma P_2 \arrow[u] & & \Sigma P_1 \arrow[ll]
\end{tikzcd}$$

$\downarrow$

cluster variable

$$\begin{tikzcd}[column sep=small, row sep=small]
 & \frac{x_1+1+x_2}{x_1 x_2} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{1+x_2}{x_1} \arrow[d, dash] & & \frac{1+x_1}{x_2} \arrow[d, dash] \\
x_2 \arrow[rr, dash] & & x_1
\end{tikzcd}$$

Thm a) $\{\text{indecom. rigid obj. of } \mathcal{C}_Q\}/\text{isom} \xrightarrow{\sim} \{\text{all cluster var. of } \mathcal{A}_Q\}$

$\Sigma P_i \mapsto x_i$

$\uparrow$ shift

b) let $T_1, \dots, T_n$ be pairwise non-isom indec. rigid obj. in $\mathcal{C}_Q$.

Then $\{\mu_1 = CC(T_1) \quad \dots \quad \mu_n = CC(T_n)\}$ form a cluster

$\Leftrightarrow \quad T = \bigoplus_{i=1}^{n} T_i$ is a cluster-tilting obj.

i.e. T is rigid & has n pairwise nonisom summand

$\mathrm{Ext}^1(T_i, T_j) = 0$

Mutation of cluster tilting sets

- Axioms of triangulated categories:

TR1 : For each morphism $u: X \to Y$, $\exists$ triangle

$$X \xrightarrow{u} Y \to Z \to \Sigma X$$

TR2: $\quad X \xrightarrow{u} Y \xrightarrow{v} Z \to \Sigma X \quad$ is a tri

$\Leftrightarrow \quad Y \xrightarrow{v} Z \to \Sigma X \to \Sigma Y \quad$ is a tri.

Suppose $T_1, \dots, T_n$ is a cluster tilting set

Q the quiver of endo. alg. of the sum of T_i

$$kQ \twoheadrightarrow \bigoplus_{i,j} \mathrm{Hom}(T_i, T_j)$$

$$e_i \longmapsto id : T_i \circlearrowleft$$

$$i \to j \longmapsto T_i \to T_j \quad \text{irr. morphism}$$

Fix k, vertex of Q

Choose $T_k \xrightarrow{u} \bigoplus_{\substack{\text{arrow} \\ k \to i}} T_i \to T_k^* \to \Sigma T_k$

& $^*T_k \to \bigoplus_{j \to k} T_j \xrightarrow{v} T_k \to \Sigma^* T_k$

These tri. are unique up to isom

& called exchange tri. associated with k & $T_1, \dots, T_n$

Thm a) $T_k^* \cong {}^*T_k$

b) $\{T_1, \dots, \hat{T_k}, \dots T_n, T_k^*\}$ is cluster tilting &

it's associated cluster is the mutation at k of the cluster associated with $T_1, \dots, T_n$

c) Two indecom L, M appears as (T_k, T_k^*) ass. with an exchange

$\Leftrightarrow \dim \mathrm{Ext}^1(L, M) = 1$

Then the exchange tri. are unique/$\cong$ non-split tri

$$L \to E \to M \to \Sigma L$$

$$M \to E' \to L \to \Sigma M$$

extend $\{\text{indecomp rep}\} \to \{\text{cluster var}\}$

$$L \mapsto X_L$$

to decom. of $\mathcal{C}_Q$ by $X_N = X_{N_1} \cdot X_{N_2}$ when $N = N_1 \oplus N_2$

Fact: if $u_1, \ldots, u_n$ is a cluster, then mutation at k.

$$\begin{array}{ccccc} u_k & u_k' & = \prod\limits_{k\to i} u_i & + & \prod\limits_{j\to k} u_j \\ \wr & \wr & \wr & & \wr \\ X_L & X_M & = X_E & + & X_{E'} \end{array}$$

Want have similar formula when $\dim \mathrm{Ext}^1(L,M) > 1$

For L, M, N obj. in $\mathcal{C}_Q$

let $\mathrm{Ext}^1(L,M)_N = \{ \varepsilon : L \to \Sigma M \mid M \to \underset{\overset{\wr\shortparallel}{N}}{E} \to L \to \Sigma M \} \subseteq \mathrm{Ext}^1(L,M)$

Thm $\mathrm{Ext}^1(L,M)_N$ is constructible in $\mathrm{Ext}^1(L,M)$.

It is union of alg. subvarieties, empty for all but finitely isom. classes of obj. N.

Thm Suppose L, M are obj. of $\mathcal{C}_Q$ s.t. $\mathrm{Ext}^1(L,M) \neq 0$

Then
$$X_L X_M = \sum_{\substack{N \\ \text{isom. class in } \mathcal{C}_Q}} \frac{\chi(\mathbb{P}\mathrm{Ext}^1(L,M)_N) + \chi(\mathbb{P}\mathrm{Ext}^1(M,L)_N)}{\chi(\mathbb{P}\mathrm{Ext}^1(L,M))} X_N$$

Acyclic case

Q: connected finite quiver without oriented cycles with vertex set $\{1, \ldots, n\}$

$k = \bar{k}$

Most result are similar as in Dynkin case

$\pi : \mathcal{D}_Q \to \mathcal{C}_Q$ triangle functor

Decomposition thm hold for $\mathcal{C}_Q$

for L in $\mathcal{C}_Q$ $\quad L = \pi(M) \oplus \bigoplus_{i=1}^{n} \pi(\Sigma P_i)^{m_i}$

take $X_L = CC(M) \cdot \prod_{i=1}^{n} x_i^{m_i}$

Thm a) $\left\{ \begin{matrix} \text{isom class of rigid} \\ \text{indecom. of } \mathcal{C}_Q \end{matrix} \right\} \xrightarrow{1-1} \{ \text{cluster alg.. } \mathcal{A}_Q \}$

$L \longmapsto X_L$

b) cluster $\Leftrightarrow$ cluster tilting set

c) for cluster tilting set $T_1, \dots, T_n$, quiver of end. alg. of $\oplus T_i$ does not have loop / 2-cycles

$\Leftrightarrow$ cluster

d) If L, M are rigid indecom. $\dim \mathrm{Ext}^1(L,M) = 1$

$X_L X_M = X_B + X_{B'}$, B, B' obtained from tri

Cluster alg	cluster category
mult.	$\oplus$
add	?
clus. var.	rigid indecom
cluster	cluster tilting sets
mutation	mutation
exchange relation	tri $\left\{ \begin{matrix} T_k \to M \to T_k^* \to \Sigma T_k \\ T_k^* \to M' \to T_k \to \Sigma T_k^* \end{matrix} \right.$