Tropicalization of cluster algebra

Def. $\mathbb{P}$ is a semifield if $\mathbb{P}$ is multiplicative abelian group endowed with addition $\oplus$: Commutative, associative & distributive

E.g. (1) Universal semifield $\mathbb{P}_{univ.}(y)$

For $y = (y)_{i\in I}$, it consists of all the rational fcn of y over $\mathbb{Q}$ with substraction-free expession

(2) Tropical semifield $\mathbb{P}_{trop}(y)$

mult. free abelian group with $\oplus$

$$\prod_{i\in I} y_i^{a_i} \oplus \prod_{i\in I} y_i^{b_i} := \prod_{i\in I} y_i^{\min(a_i, b_i)}$$

$$\pi_{trop} : \mathbb{P}_{univ.}(y) \to \mathbb{P}_{trop}(y)$$
$$y_i \mapsto y_i$$
$$\mathbb{Q}_{>0} \ni c \mapsto 1$$

Cluster alg w/. coeff

initial seed : (B, x, y)

coeff. semifield: $\mathbb{P}_{univ}(y)$

Fix $D = diag(d_i)$ $\{d_i\}$ coprime

$D = I$ if B is skew sym

other seed: (B', x', y')

- B' – DB skew-sym
- $x' = (x_i')_{i\in I}$ $\quad x_i' \in \hat{\mathbb{Q}}(x)$ $\quad \hat{\mathbb{Q}} := \mathbb{Q}\mathbb{P}_{univ.}(y)$ fractional field of $\hat{\mathbb{Z}} = \mathbb{Z}\mathbb{P}_{univ}(y)$
- $y' = (y_i')_{i\in I}$ $\quad y_i' \in \mathbb{P}_{univ}(y)$

mutation at k: $\mu_k(B', x', y') = (B'', x'', y'')$

$$b''_{ij} = \begin{cases} -b'_{ij} & i \text{ or } j = k \\ b'_{ij} + [-b'_{ik}]_+ b'_{kj} + b'_{ik}[b'_{kj}]_+ & i,j \neq k \end{cases}$$

$$y''_i = \begin{cases} {y'_k}^{-1} & i = k \\ y'_i \dfrac{(1 \oplus y'_k)^{[-b'_{ki}]_+}}{(1 \oplus {y'_k}^{-1})^{[b'_{ki}]_+}} & i \neq k \end{cases}$$

$$x''_i = \begin{cases} {x'_k}^{-1} \left(\dfrac{1}{1 \oplus {y'_k}^{-1}} \prod_{j \in I} {x'_j}^{[b'_{jk}]} + \dfrac{1}{1 \oplus y'_k} \prod_{j \in I} {x'_j}^{[-b'_{jk}]_+} \right) & i = k \\ x'_i & i \neq k \end{cases}$$

where $[x]_+ = \max(x, 0)$

Define $\hat{y}'_i = y'_i \prod_{j \in I} {x'_j}^{b'_{ji}}$

$$\hat{y}''_i = \begin{cases} {\hat{y}'_k}^{-1} & i = k \\ \hat{y}'_i \dfrac{(1 + \hat{y}'_k)^{[-b'_{ki}]_+}}{(1 + {\hat{y}'_k}^{-1})^{[b'_{ki}]_+}} & i \neq k \end{cases}$$

ε-expression

$$\Rightarrow b''_{ij} = \begin{cases} -b'_{ij} & i \text{ or } j = k \\ b'_{ij} + [-\varepsilon b'_{ik}]_+ b'_{kj} + b'_{ik}[\varepsilon b'_{kj}]_+ & i,j \neq k \end{cases}$$

$\varepsilon = \{1, -1\}$

Indep. of choice of ε since $[-\varepsilon b'_{ik}] = \dfrac{\varepsilon b_{kj} + |b_{kj}|}{2}$

plug in & check.

Similarly,

$$y_i'' = \begin{cases} y_k'^{-1} & i=k \\ y_i' y_k'^{[\varepsilon b_{ki}']_+} (1 \oplus y_k'^{\varepsilon})^{-b_{ki}'} & i \neq k \end{cases}$$

$$x_i'' = \begin{cases} x_k'^{-1} \left(\prod_{j \in I} x_j'^{[-\varepsilon b_{kj}']_+} \right) \frac{1+\hat{y}_k'^{\varepsilon}}{1 \oplus y_k'^{\varepsilon}} & i=k \\ x_i' & i \neq k \end{cases}$$

Thm (Separation formula)

For each seed (B', x', y') of any cluster alg. $\mathcal{A}(B, x, y)$

$\exists$ poly $F_i'(y)$ call F-poly

Interger matrix $C' = (c_{ij}')_{i,j \in I}$ C-matrix

" $G' = (g_{ij}')_{i,j \in I}$ G-matrix

s.t.

$$y_i' = \left(\prod_{j \in I} y_j^{c_{ji}'} \right) \prod_{j \in I} F_j'(y)_{\oplus}^{b_{ji}'}$$

$$x_i' = \left(\prod_{j \in I} x_j^{g_{ji}'} \right) \frac{F_i'(\hat{y})}{F_i(y)_{\oplus}}$$

$F_i(y)_{\oplus}$: replace + in substraction free of $F_i(y)$ w. $\oplus$ in $\mathbb{P}_{univ}(y)$

Column of C' & G' are called c-vectors, g-vectors

Mark, ..., proved the sign-coherence conjecture!

i.e. Every column of a C-matrix (i.e. every c-vector) is a nonzero vector & its nonzero components are either all positive or all negative.

With sign-coherent, assign the sign $\varepsilon_k' \in \{-1, 1\}$ defined by the sign of the k-th column of the C-matrix of (B', x', y').

Call ε_k' the tropical sign of (B', x', y') at $k \in I$.

Tropicalization!

Write $[\cdot] = \Pi_{trop}(\cdot)$

For $y_i' \in \mathbb{P}_{univ}(y)$, we call $[y_i'] \in \mathbb{P}_{trop}(y)$ the tropical y-var.

<u>Thm</u> $$[y_i'] = \prod_{j \in I} y_j^{c_{ji}'}$$

$$[F_i'(y)]_{\oplus} = 1$$

So in seperation formula

$$y_i' = \underbrace{\left(\prod_{j \in I} y_j^{c_{ji}'}\right)}_{\text{trop part}} \underbrace{\prod_{j \in I} F_j'(y)_{\oplus}^{b_{ji}'}}_{\text{non-trop part}}$$

Set $\varepsilon = \varepsilon_k'$.

Then $[1 \oplus y_k'^{\varepsilon_k'}] = 1$

So

$$y_i'' = \begin{cases} y_k'^{-1} & i = k \\ \underbrace{y_i' y_k'^{[\varepsilon b_{ki}']_+}}_{\text{trop}} \underbrace{(1 \oplus y_k'^{\varepsilon})^{-b_{ki}'}}_{\text{non-trop}} & i \neq k \end{cases}$$

$$[y_i''] = \begin{cases} [y_k']^{-1} & i = k \\ [y_i'][y_k']^{[\varepsilon_k' b_{ki}']_+} & i \neq k \end{cases}$$

X-variable:

Set $\varepsilon = \varepsilon_k'$

$[1 \oplus y_k'^{\varepsilon_k'}] = 1$

Can think of $[x_i] = x_i$ & $[\hat{y}_i] = \hat{y}_i$ are indep. variable and exchange relation for $[x_i']$ and $[\hat{y}_i']$ are given by

$$\hat{y}_i'' = \begin{cases} \hat{y}_k'^{-1} & i = k \\ \hat{y}_i' \dfrac{(1+\hat{y}_k')^{[-b_{ki}']_+}}{(1+\hat{y}_k'^{-1})^{[b_{ki}']_+}} & i \neq k \end{cases}$$

Make 'second tropicalization'

$$[\![\cdot]\!] : \mathbb{P}_{univ}(\hat{y}) \longrightarrow \mathbb{P}_{trop}(\hat{y})$$
$$[x_i'] \longmapsto [\![x_i']\!]$$

call y-tropical x-variable

$$\therefore [\![x_i']\!] = \prod_{j \in I} x_j^{g_{ji}'}$$

$$[\![x_i'']\!] = \begin{cases} [\![x_k']\!]^{-1} \prod_{j \in I} [\![x_j']\!]^{[-\varepsilon_k' b_{jk}']_+} & i = k \\ [\![x_i']\!] & i = k \end{cases}$$

Prop: $(G'D^{-1})^T = (DC') = I$

col. vectors of DC' & $G'D^{-1}$ are dual basis of $\mathbb{Q}^n$ to each other.