1 Problems II: Sets and Functions (Page 117-118)

11. Give a proof or a counterexample of the following statements:
   (vi) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy \geq 0; \)
   (x) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0 \) and \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0 \).

A) (vi) Choose \( y = 0 \). Then for any \( x \in \mathbb{R}, xy = 0 \geq 0 \).
   Alternatively, one can put \( y = x \).
(x) Let P denote the statement \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0 \), and let Q denote the statement \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0 \).
   We have to find whether the statement \( (P \text{ and } Q) \) is true or not.
   If \( x \in \mathbb{R} \), then putting \( y = 1 - x \), we see that statement P is true.
   If \( x \in \mathbb{R} \), then putting \( y = -x \), we see that statement Q is true.
   Since both P and Q are true, the statement \( (P \text{ and } Q) \) is true.

14. Define functions \( f \) and \( g \): \( \mathbb{R} \to \mathbb{R} \) by \( f(x) = x^2 \) and \( g(x) = x^2 - 1 \). Find the functions \( fof, fog, gof, gog \).
   List the elements of the set \( \{ x \in \mathbb{R} | fg(x) = gf(x) \} \).

A) \( fof : \mathbb{R} \to \mathbb{R} \) is given by \( fof(x) = f(f(x)) = f(x^2) = (x^2)^2 = x^4 \).
   \( fog : \mathbb{R} \to \mathbb{R} \) is given by \( fog(x) = f(g(x)) = f(x^2 - 1) = (x^2 - 1)^2 = x^4 - 2x^2 + 1 \).
   \( gof : \mathbb{R} \to \mathbb{R} \) is given by \( gof(x) = g(f(x)) = g(x^2) = (x^2)^2 - 1 = x^4 - 1 \).
   \( gog : \mathbb{R} \to \mathbb{R} \) is given by \( gog(x) = g(g(x)) = g(x^2 - 1) = (x^2 - 1)^2 - 1 = x^4 - 2x^2 \).

If \( x \) is such that \( fg(x) = gf(x) \), then \( x^4 - 2x^2 + 1 = x^4 - 1 \)
   \( \implies 2x^2 = 2 \implies x = \pm 1 \).
   Therefore, \( \{ x \in \mathbb{R} | fg(x) = gf(x) \} = \{-1, +1\} \).

16. Determine which of the following functions \( f_i : \mathbb{R} \to \mathbb{R} \) are injective, which are surjective and which are bijective.
   Write down an inverse function of each of the bijections.
   (i) \( f_1(x) = x - 1 \);
(i) \( f_1(x) = x^3; \)
(ii) \( f_2(x) = x^3 - x; \)
(iii) \( f_3(x) = x^3; \)
(iv) \( f_4(x) = x^3 - 3x^2 + 3x - 1; \)
(v) \( f_5(x) = e^x; \)
(vi) \( f_6(x) = \begin{cases} 
  x^2 & \text{if } x \geq 0, \\
 -x^2 & \text{if } x \leq 0.
\end{cases} \)

A) (i) Define \( g_1 : \mathbb{R} \to \mathbb{R} \) by \( g_1(x) = x + 1. \) Then \( f_1 \circ g_1(x) = f_1(g_1(x)) = f_1(x + 1) = x + 1 - 1 = x \) \( \forall x \in \mathbb{R}. \) Also \( g_1 \circ f_1(x) = g_1(f_1(x)) = g_1(x - 1) = x - 1 + 1 = x \) \( \forall x \in \mathbb{R}. \) Therefore, \( g_1 = f_1^{-1}, \) which implies \( f_1 \) is invertible, and hence bijective, and subsequently injective and surjective.

(ii) Define \( g_2 : \mathbb{R} \to \mathbb{R} \) by \( g_2(x) = x^{1/3}. \) Then \( f_2 \circ g_2(x) = f_2(g_2(x)) = f_2(x^{1/3}) = (x^{1/3})^3 = x \) \( \forall x \in \mathbb{R}. \) Also \( g_2 \circ f_2(x) = g_2(f_2(x)) = g_2(x^3) = (x^3)^{1/3} = x \) \( \forall x \in \mathbb{R}. \) Therefore, \( g_2 = f_2^{-1}, \) which implies \( f_2 \) is invertible, and hence bijective, and subsequently injective and surjective.

(iii) Since \( f_3(0) = f_3(1) = 0, \) \( f_3 \) is not injective, and hence not bijective also. Now let \( y \) be any real number. Look at the function \( f \) given by \( f(x) = x^3 - x - y. \) Note that \( f \) is a continuous function on \( \mathbb{R}. \) Note that when \( x \) is a very negative number, then \( f(x) \) is negative. Similarly, when \( x \) is a large positive number, \( f(x) \) will be positive. Then, by the Intermediate Value Theorem in Calculus, \( \exists x \) such that \( f(x) = 0, \) that is, there exists \( x \) such that \( x^3 - x = y. \) Therefore, \( f_3 \) is surjective.

Note: This question is quite tricky as it uses a lot of calculus. It will not be on the test.

(iv) Note that \( x^3 - 3x^2 + 3x - 1 = (x - 1)^3 = f_2(x - 1) = f_2(f_1(x)) = f_2 \circ f_1(x). \) So, \( f_4 = f_2 \circ f_1. \) Since both \( f_2 \) and \( f_1 \) are bijective, \( f_4 \) is also bijective, and hence also surjective and injective. \( f_4^{-1} = f_1^{-1} \circ f_2^{-1}. \)

(v) Since there does not exist a real number \( x \) such that \( e^x = 0, \) \( f_5 \) is not surjective, and hence not bijective. Now suppose that \( f_5 \) is not injective. Then \( \exists x_1, x_2 \) distinct in \( \mathbb{R} \) such that \( e^{x_1} = e^{x_2}. \) \( \implies e^{x_1-x_2} = 1 \implies x_1 - x_2 = 0. \) This gives a contradiction as \( x_1 \) and \( x_2 \) are distinct. Therefore \( f \) is injective.

(vi) Define \( g_6 : \mathbb{R} \to \mathbb{R} \) by
\[
g_6(x) = \begin{cases} 
  \sqrt{x} & \text{if } x \geq 0, \\
 -\sqrt{-x} & \text{if } x \leq 0.
\end{cases}
\]
Let \( x \geq 0. \) Then, \( f_6 \circ g_6(x) = f_6(g_6(x)) = f_6(\sqrt{x}) = (\sqrt{x})^2 = x \) as \( \sqrt{x} \geq 0. \)
Let \( x \leq 0. \) Then, \( f_6 \circ g_6(x) = f_6(g_6(x)) = f_6(-\sqrt{-x}) = -(-\sqrt{-x})^2 = x \) as \( -\sqrt{-x} \leq 0. \)

So \( f_6 \circ g_6(x) = x \) \( \forall x \in \mathbb{R} \)
Let \( x \geq 0. \) Then, \( g_6 \circ f_6(x) = g_6(f_6(x)) = g_6(x^2) = \sqrt{x^2} = x. \)
Let \( x \leq 0 \). Then, \( g_6 \circ f_6(x) = g_6(f_6(x)) = g_6(-x^2) = -\sqrt{-(-x^2)} = -(x) = x \), as \( \sqrt{x^2} = -x \) when \( x \leq 0 \).

So \( g_6 \circ f_6(x) = x \) \( \forall x \in \mathbb{R} \).

Therefore \( g_6 = f_6^{-1} \), which implies \( f_6 \) is invertible, and hence bijective, and subsequently injective and surjective.

2 Problems III: Numbers and Counting (Page 182-185)

1. Of the 182 students who are taking three first year core Mathematics modules (Reasoning, Algebra and Calculus), 129 like Reasoning, 129 like Algebra, 129 like Calculus, 85 like Reasoning and Algebra, 89 like Reasoning and Calculus, 86 like Algebra and Calculus, and 54 like all three modules. How many of the students like none of the core modules?

A) Let \( R \) denote the set of students who like Reasoning, \( A \) denote the set of students who like Algebra, \( C \) denote the set of students who like Calculus.

Then \( |R| = |A| = |C| = 129 \).

\[ |A \cap R| = 85, \ |C \cap R| = 89, \ |A \cap C| = 86, \ |A \cap R \cap C| = 54 \]

Then, by the inclusion-exclusion principle for \( A, C \) and \( R \),

\[ |A \cup C \cup R| = |A| + |C| + |R| - |A \cap C| - |A \cap R| - |C \cap R| + |A \cap R \cap C| = 129 + 129 + 129 - 86 - 85 - 89 + 54 = 1 \]

The set of students who do not like any of the core modules is \( A^c \cap C^c \cap R^c \).

\[ |A^c \cap C^c \cap R^c| = \text{Total number of students} - |A \cup C \cup R| = 182 - 181 = 1 \]

So, only one student does not like any of the core modules.

2. Of the 170 students who took all the first year core modules last year, 124 liked Reasoning, 124 liked Algebra, 124 liked Calculus, 10 liked only Reasoning, nobody liked only Algebra, 4 liked only Calculus and 2 liked none of the modules. How many students liked all three core modules?

A) Let \( R \) denote the set of students who liked Reasoning, \( A \) denote the set of students who liked Algebra, \( C \) denote the set of students who liked Calculus.

Then \( |R| = |A| = |C| = 124 \).

The set of students who liked only Reasoning is given by \( R \cap A^c \cap C^c \). Therefore,

\[ |R \cap A^c \cap C^c| = 10 \]
14. For \( n \in \mathbb{Z}^+ \), suppose that \( A \subseteq N_{2n} \) and \( |A| = n+1 \). Prove that \( A \) contains a pair of distinct integers \( a_b \) such that \( a_b \) divides \( b \).

A) Define \( f : A \rightarrow \{1, 3, 5, \ldots, 2n-1\} \) by \( f(a) = \) the greatest odd integer which divides \( a \).

By the Pigeonhole Principle (Theorem 11.1.2, Page 136), \( f \) cannot be an injection as \( |A| = n+1 \geq n = |\{1, 3, 5, \ldots, 2n-1\}| \).

So, \( \exists a_1, a_2 \) distinct elements in \( A \) such that \( f(a_1) = f(a_2) \).

Note that \( \frac{a_1}{f(a_1)} \) cannot have any odd factors other than 1, as otherwise, we could multiply that odd factor to \( f(a_1) \) and get an odd factor of \( a_1 \) larger than \( f(a_1) \).

So, \( \frac{a_1}{f(a_1)} \) has to be a power of 2. \( \implies a_1 = 2^{c_1} f(a_1) \) for some \( c_1 \geq 0 \).

Similarly, \( a_2 = 2^{c_2} f(a_2) \) for some \( c_2 \geq 0 \).

\( a_2 = 2^{c_2} f(a_1) \) as \( f(a_1) = f(a_2) \).

If \( c_1 > c_2 \), then \( a_2 \) divides \( a_1 \) and so we have obtained two distinct integers in \( A \) such that one divides the other.

If \( c_2 > c_1 \), then \( a_1 \) divides \( a_2 \) and again we have obtained two distinct integers in \( A \)
such that one divides the other.
So in any case, we have obtained two distinct integers in \( A \) such that one divides the other.

20. Use the pigeonhole principle to prove that, given ten distinct positive integers less than 107, there exist two disjoint subsets with the same sum.

A) Let us denote by \( A \) the set of the given ten distinct positive integers less than 107. Since the largest element in \( A \) is less than or equal to 106 and the second largest is less than or equal to 105 and so on, we can see that the sum of elements of \( A \) is less than or equal to 106 + 105 + 104 + ..... + 97 = 1015.
So the sum of elements of any non-empty subset of \( A \) would also be less than or equal to 1015. Since \( A \) has \( 2^{10} - 1 = 1023 \) non-empty subsets and there are only 1015 possibilities for the sum of the elements of a non-empty subset, by the pigeonhole principle, there exist two distinct non-empty subsets of \( A \), say \( A_1 \) and \( A_2 \), such that the sum of elements of \( A_1 \) is equal to the sum of elements of \( A_2 \).
Now, if \( a \) is an element that belongs to both \( A_1 \) and \( A_2 \), then removing \( a \) from both \( A_1 \) and \( A_2 \) reduces the sum of elements of \( A_1 \) and \( A_2 \) by \( a \). So we can remove all the elements common to \( A_1 \) and \( A_2 \) and obtain two disjoint sets with the same sum.
Note: Once can also define the function \( f : \mathcal{P}(A) - \{\phi\} \rightarrow \{1, 2, 3, \ldots, 1015\} \) by
\[
f(\text{any non-empty subset of } A) = \text{sum of elements of that subset}
\]
and then apply Theorem 11.1.2, where \( \mathcal{P}(A) \) denotes the power set of \( A \).