Chapter 1
1.1 Mathematical Statements

The first step into the mathematical world is to learn its language. The life of mathematical language is its rigorousness.

A mathematical sentence is usually called a mathematical statement. Propositions are a special kind of mathematical statements. A proposition is a sentence which is either true or false. Let’s consider some examples:

(i) $1 + 2 = 3$
(ii) $x + 1 > x$
(iii) $\pi = 3$

The statements (i) (ii) are clearly true, while (iii) is clearly false. They are all propositions. Propositions are statements that we can always determine whether they are true or false (if we are smart enough). Two more examples:

(iv) 12 may be written as the sum of two prime numbers.
(v) Every even integer greater than 2 may be written as the sum of two prime numbers.

It takes a bit work to check (iv): $12 = 5 + 7$, so it is true, and thus is a proposition.

It is very difficult to determine whether (v) is true or false. Indeed, this is a well-known open question, called the Goldbach Conjecture. However, it has to be either true or false: Every even integer greater than 2, either can be or cannot be, written as the sum of the prime numbers. There is no third choice. So (v) is also a proposition. Now let’s take a look at the following:

(vi) $n^2 - 2n > 0$.
(vii) $\pi$ is a special number.
(viii) Math 109 is a great class.
(ix) $2 + 3$
(x) $\pi$ is not a rational number.

None of (vi)-(ix) is a proposition. We cannot tell if (vi) is true or not, as we are not given the values of $n$. It become a proposition if we reformulate it in a more rigorous way:
(a) Let $n$ be any integer, then $n^2 - 2n > 0$;
(b) Let $n = 3$, then $n^2 - 2n > 0$.
   (a) is false, (b) is true. They are both propositions.

(vii) is NOT a proposition, as we are not told what “special” means. In other words, one has to define “special” to make it rigorous:
(c) We call a number $x$ special if $\sin x = 0$. Then $\pi$ is special. (T)
(d) We call a number $x$ special if $x^2$ is an integer. Then $\pi$ is special. (F)

Compare (vii) to (x). (x) is a proposition. The reason is “rational” has a well-known convention in mathematics. Its meaning is taken for granted. (x) is true.

(viii) is not a proposition. Math 109 may mean different courses in different universities. Also, it does not specify the meaning of “great”.
(ix) is not even a sentence, so it’s not a proposition.

Sentences like (vi) are called predicates. They become propositions if we assign values to the symbols in them (like “n” in (vi)). These symbols are called free variables. The word “statement” will be used to denote either a proposition or a predicate.
1.2 Logical Connective

There are some logical connectives to build up complicated statement out of simpler ones: “or”, “and”, “not”.

(1) The connective “or”

Example: For integers $a$, $ab = 0$ if $a = 0$ or $b = 0$.

The statement “a=0 or b=0” is true if $a = 0$ and is also true if $b = 0$. Note $a = 0$ and $b = 0$ can be both true. In this case, “a=0 or b=0” is clearly also true. This is called the “inclusive” use of “or”. In general, let $P,Q$ be two statements. Then “$P$ or $Q$” means either $P$ holds or $Q$ holds (or both holds). “$P$ or $Q$” is called the disjunction of $P$ and $Q$.

Truth table for “or”:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P or Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

In daily life, “or” is often used in the exclusive sense. Example: “Everyone will travel there by bus or by train.” This would normally be taken to mean that everyone uses one or the other form of transportation but not both.

Example: $a \leq b$ means $a < b$ or $a = b$;

$a = \pm b$ means $a = b$ or $a = -b$;

Q: Are the following propositions true?
(i) $1 \leq 2$ (ii) $2 \leq 2$ (iii) $1 = \pm 1$.

They are all true.

(2) The connective “and”

Let $P$, $Q$ be two statements. “$P$ and $Q$” means $P$ holds and $Q$ holds. “$P$ and $Q$” is called the conjunction of $P,Q$. 
Truth table for “and”:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P and Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
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<td>F</td>
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Example: $3 < \pi < 4$ means $\pi > 3$ and $\pi < 4$.

(3) The connective “not”

Let P be a statement. “Not P” means P does not hold. It is called the negation of P.

Truth table:

<table>
<thead>
<tr>
<th>P</th>
<th>Not P</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Example:
(a) P: $x > y$
   not P: $x \leq y$
(b) P: $1 = \pi$
   not P: $1 \neq \pi$

Q: Let $f(x)$ be the polynomial $x^3 - x^2 - x$. Let P be the statement: For real number $a$, if $f(a) = 0$, then $a > 0$. Find “not P”. Is it true?
( Hint: “P” says “whenever $f(a) = 0$, then $a > 0$”. “not P” should be “There is a situation when $f(a) = 0$ satisfied, $a > 0$ does not hold (i.e. $a \leq 0$)”.

Solution: Not P: There exists $a \leq 0$ for which $f(a) = 0$.

Note that $f(x) = x(x^2 - x - 1) = x(x - \frac{1 + \sqrt{5}}{2})(x - \frac{1 - \sqrt{5}}{2})$. So $f(x) = 0$ only when $x = 0$ or $x = \frac{1 + \sqrt{5}}{2}$ or $x = \frac{1 - \sqrt{5}}{2}$. So P is false. Not P is true.