Chapter 12

Recall:

**Proposition:** Suppose $|X| = m > 0, |Y| = n > 0$. The number of functions $X \rightarrow Y$ is $n^m$.

**Definition:** Given sets $X$ and $Y$. We denote the set of functions from $X$ to $Y$ by $\text{Fun}(X, Y)$.

Hence, given $|X| = m, |Y| = n$, $|\text{Fun}(X, Y)| = n^m$.

**Example.** Let $X = \{a\}, Y = \{b_1, b_2\}$, then $\text{Fun}(X, Y) = \{f_1, f_2\}$, where
- $f_1 : f_1(a) = b_1$
- $f_2 : f_2(a) = b_2$.

**Example.** Let $X = \{a_1, a_2\}, Y = \{b_1, b_2\}$, then $\text{Fun}(X, Y) = \{f_1, f_2, f_3, f_4\}$, where
- $f_1 : f_1(a_1) = b_1, f_2(a_2) = b_2$
- $f_2 : f_1(a_1) = b_1, f_2(a_2) = b_2$
- $f_3 : f_1(a_1) = b_2, f_2(a_2) = b_1$
- $f_4 : f_1(a_1) = b_2, f_2(a_2) = b_2$

**Proposition:** Given $|X| = m, |Y| = n$, then the number of injections $X \rightarrow Y$ is given by $n(n-1)\cdots(n-m+1)$.

**Definition:** The set of injections from $X$ to $Y$ is denoted by $\text{Inj}(X, Y)$.

Hence, given $|X| = m, |Y| = n$, $|\text{Inj}(X, Y)| = n(n-1)\cdots(n-m+1)$.

**Definition:** Given a set $X$, a bijection $X \rightarrow X$ is called a permutation of the set $X$.

**Corollary:** Given $|X| = n$, the number of permutations of $X$ is given by $n! = n(n-1)(n-2)\cdots1$. (‘$n!’ is read as ‘factorial’.)

1 **Counting subsets**

Recall:

$$P(X) = \{A| A \subseteq X\}$$

**Proposition:** Given $|X| = n$, the power set $P(X)$ is a cardinality of $2^n$. That is,

$$|P(X)| = 2^n.$$
Definition: Given a non-negative integer $r$, and $|X| = n > r$, an $r$-subset of $X$ is a subset of $X$ with $r$ elements. We will denote the set of $r$-subsets of $X$ by $P_r(X)$, i.e.,

$$P_r(X) = \{A \subseteq X \mid |A| = r\}$$

We define $\binom{n}{2}$ to be $P_r(X)$ with $|X| = n$, $\binom{n}{2}$ is read as ‘$n$ choose $r$’.

Eg. Let $X = \{a, b, c, d\}$, then:

$$P_0(X) = \{\emptyset\}.$$

$$P_1(X) = \{\{a\}, \{b\}, \{c\}, \{d\}\}$$

$$P_2(X) = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$$

$$P_3(X) = \{\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$$

$$P_4(X) = X$$

$$P_r(X) = \emptyset \text{ if } r > 4.$$  

Thus, $\binom{4}{0} = 1, \binom{4}{1} = 4, \binom{4}{2} = 6, \binom{4}{3} = 4, \binom{4}{4} = 1, \binom{n}{r} = 0 \text{ if } r > 4.$

Proposition: (1) $\binom{n}{r} = 0$ if $r > n$.

(2) $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{n} = 1$.

(3) $\binom{n}{r} = \binom{n}{n-r}$ for $0 \leq r \leq n$.

Proposition: Let $n \in \mathbb{Z}^+$, $\sum_{i=0}^{n} \binom{n}{i} = 2^n$. 

2
2 Evaluating binomial coefficients

Theorem: Let \( n, r \) be integers with \( 0 \leq r \leq n \). Then \( \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!} \)

Proposition: Let \( 1 \leq r \leq n \), then \( \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \).

Proof: 
\[
\text{RHS} = \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-1-r)!} \\
= \frac{r(n-1)!+(n-1)!(n-r)}{r!(n-r)!} \\
= \frac{(n-1)!(r+n-r)}{r!(n-r)!} \\
= \frac{n!}{r!(n-r)!} = LHS.
\]

Theorem: (The binomial theorem) Let \( a, b \in \mathbb{R}, n \in \mathbb{Z}_{\geq 0} \). Then
\[
(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.
\]

E.g. What is the coefficient of \( a^6 b^4 \) in the expansion of \((a + b)^{10}\)?

The coefficient is given by \( \binom{10}{6} = \binom{10}{4} = \frac{10!}{6!4!} = \frac{10\cdot9\cdot8\cdot7}{1\cdot2\cdot3\cdot4} = 210 \).