In this chapter we will discuss indirect ways of proof: proof by contradiction and proof by contrapositive.

1 Proof by contradiction

A template for proof by contradiction.

Question: Prove a statement $P$.

Proof: Suppose, for contradiction, that the statement $P$ is false.

Then (Present some argument leading to a contradiction).

Hence our assumption that ‘$P$ is false’ cannot be true.
We thus have proved $P$ must be true as required.

Remark: When we use proof by contradiction? A typical case is when the statement $P$ is a negative statement. That is, the statement contains a phrase like ‘not exist’, ‘not equal’, etc.

Eg: There do not exist integers $m$ and $n$ such that $14m + 20n = 101$.

Proof: Suppose, by contradiction, that the statement is false. That is, suppose there exist integers $m$ and $n$ such that

$$14m + 20n = 101. \quad (*)$$

Then the LHS of $(*)$ equals to $2(7m + 10n)$, which is an even integer. The RHS of $(*)$, however, is an odd integer. They cannot be equal, we thus get a contradiction! Hence our assumption at the beginning cannot be true. We thus have proved that there are no such $m$ and $n$. 
Eg: Prove that for any real numbers $a$ and $b$, $2 + a^2 + b^2 \neq 1 + 2ab$.

Proof: Suppose, by contradiction, that for some real numbers $a$ and $b$, $2 + a^2 + b^2 = 1 + 2ab$. That is, $(2 + a^2 + b^2) - (1 - 2ab) = 0$. Then we have $1 + (a^2 + b^2 - 2ab) = 0$. This yields

$$1 + (a - b)^2 = 0 \quad (\ast).$$

Note the LHS of $(\ast): 1 + (a - b)^2 \geq 1$. It cannot equal to zero, which is the RHS of $(\ast)$. This is a contradiction. Hence our assumption at the beginning cannot be true. We thus have proved the desired statement.

Exercise: Let $f(x) = \frac{2x + 3}{x + 2}$. Prove for any real number $x$, $f(x) \neq 2$.

2 Proof by contrapositive

Now we consider how to prove an implication: ‘$P \Rightarrow Q$’. The idea is similar to proof by contradiction: We assume $P$, and suppose $Q$ is false, then present some argument leading to the conclusion that $P$ is not true. This is a contradiction. But we have a more elegant way to write down the idea above, called ‘proof by contrapositive’.

Remark: The statement ‘$P \Rightarrow Q$’ is equivalent to its contrapositive ‘$\neg Q \Rightarrow \neg P$’.

Template of proof by contrapositive.

Question: Prove $P \Rightarrow Q$

Proof: We will prove the contrapositive of the desired statement: [State the contrapositive here]. Then (give a proof of this contrapositive.)

Eg. Let $m$ be an integer. Prove: $m^2$ is even $\Rightarrow$ $m$ is even.

Proof: We will prove the contrapositive of the desired statement: $m$ is odd $\Rightarrow$ $m^2$ is odd. Then Copy the proof from Eg.3 in note of Chapter 2-3.
Eg. If $a$ and $b$ are real numbers, then $ab = 0 \Rightarrow a = 0$ or $b = 0$.

Proof: We will prove the contrapositive of the desired statement: $a \neq 0$ and $b \neq 0 \Rightarrow ab \neq 0$. This is clearly true, so we have established the desired statement.

Eg. Let $a$ be a real number. Prove: $a^3 \geq 2a^2 \Rightarrow a = 0$ or $a \geq 2$.

Proof: We will prove the contrapositive of the desired statement: $a \neq 0$ and $a < 2 \Rightarrow a^3 < 2a^2$. Now assume $a \neq 0$ and $a < 2$. Then note $a^3 - 2a^2 = a^2(a - 2)$. Moreover, $a^2 > 0$, $a - 2 < 0$. Thus $a^2(a - 2) < 0$. This implies $a^3 - 2a^2 < 0$, i.e., $a^3 < 2a^2$. We have thus proved the contrapositive statement, and the original statement is also true.

Remark: To prove $A < B$, most time we verify $A - B < 0$. 
