Mathematical induction is often used to prove a statement $P(n)$ that has a free variable $n$, where $n$ is an integer.

Template for proof by induction:

**Question:** Prove $P(n)$.

**Proof:** We use induction on $n$.

Basic step: Prove the statement for the initial $n$ (most times it is $n = 1$.)

Inductive step: Suppose the statement is true when $n = k$. That is, we suppose [State the statement $P(k)$]. Then [Use the above inductive hypothesis $P(k)$ to derive that $P(k+1)$ is true.] This proves the inductive step. Hence, by induction, the statement is true for all $n$.

Eg.1. Use mathematical induction to prove for any $n \geq 1, n \leq 2^n$.

Proof: We use induction on $n$.

Basic step: (Verify $P(1)$)

When $n = 1, 2^n = 2$. Note $1 \leq 2$. Thus the statement is true when $n = 1$.

Inductive step: Suppose the statement is true when $n = k$. That is, we suppose $k \leq 2^k$. Then when $n = k + 1, k + 1 \leq 2^k + 1 \leq 2^k + 2^k = 2^{k+1}$; Thus $k + 1 \leq 2^{k+1}$. This proves $P(k+1)$ and finishes the inductive step. Hence, by induction, the statement is true for all $n \geq 1$. 

Eg.2: Prove for any integer \( n \geq 4 \), \( n^2 \leq 2^n \).

Proof: We use induction on \( n \)

Basic step: (Prove P(4))

When \( n = 4 \), 16 = \( n^2 \) = \( 2^n \) = 16, thus the statement is true for \( n = 4 \).

Inductive step:

Suppose the statement is true when \( n = k \) for some \( k \geq 4 \). That is, \( k^2 \leq 2^k \). Then we consider the case when \( n = k+1 : (k+1)^2 = k^2 + 2k + 1 \leq 2^k + 2k + 1 \leq 2^k + 3k \leq 2^k + k^2 \leq 2^k + 2^k = 2^{k+1} \). Thus \( (k+1)^2 \leq 2^{k+1} \). This proves \( P(k+1) \) and finishes the inductive step. Hence, by induction, the statement is true for all \( n \geq 4 \).

Excercise: Prove for any \( n \geq 4 \), \( n! > 2^n \).

Eg.3: Prove for any \( n \geq 1 \), \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \).

Proof: We use induction on \( n \)

Basic step:

When \( n = 1 \), LHS = 1 = 1; RHS = \( \frac{1 \times 2 \times 3}{6} = 1 \). They are equal.

Inductive step:

Suppose the statement is true when \( n = k \). That is, we suppose \( \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6} \).

Now we consider the case when \( n = k+1 : \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(2k^2 + 7k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} \). This proves \( P(k+1) \) and finishes the inductive step.

Hence, by induction, the statement is true for all \( n \geq 1 \).

Eg.4: Prove for any \( n \geq 1 \), \( 6^{n+1} + 7^{2n-1} \) is divisible by 43.

Proof: We use induction on \( n \).

Basic step:

When \( n = 1 \), \( 6^{n+1} + 7^{2n-1} = 6^2 + 7 = 43 \). It’s divisible by 43.

Inductive step:

Suppose the statement is true when \( n = k \). That is, \( 6^{k+1} + 7^{2k-1} \) is divisible by 43. Then when \( n = k + 1 : 6^{n+1} + 7^{2n-1} = 6^{k+2} + 7^{2(k+1)-1} = 6^{k+2} + 7^{2k+1} = 6 \times 6^{k+1} + 49 \times 7^{2k-1} = 6 \times (6^{k+1} + 7^{2k-1}) + 43 \times 7^{2k-1} \). By inductive hypothesis, the first term is divisible by 43; the second term is a multiple of 43. Thus the whole quantity is also divisible by 43. This proves \( P(k+1) \) and finishes the inductive step. Hence, by induction, the statement is true for all \( n \geq 1 \).