1 Functions

Definition: Suppose that $X, Y$ are sets. A function (or map, mapping) from $X$ to $Y$ is the assignment of a unique element of $Y$ to each element of $X$. If $f$ is a function from $X$ to $Y$ we write $f : X \to Y$. If the function $f$ assigns $y \in Y$ to $x \in X$, we write $y = f(x)$.

Remark: The element $y = f(x)$ in $Y$ is called the image of $x$ under $f$ (or the value of $f$ at $x$). The element $x$ is called the pre-image of $y$. The set $X$ is called the domain of the function $f$. The set $Y$ is called the codomain.

Question: How to describe a function?

(1). Describe by table.
E.g: See page 90 in the textbook.

(2). Describe by graph.
E.g: See page 90 in the textbook.

(3). Describe by formula.
E.g: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 + 1$.

Question: Let $X$ be a set with 3 elements, $Y$ a set with 2 elements. How many functions are there from $X$ to $Y$?

Question: Let $A$ be a set with 2 elements. How many functions are there from $A$ to $A$?

Remark: Let $X$ be a set with $m$ elements, $Y$ a set with $n$ elements. There are $m^n$ functions from $X$ to $Y$.
**Remark:** Sometimes a function is given by a formula and the domain (and codomain) are not specified. In this case, we always understand the domain as the subset of \( \mathbb{R} \) where the formula makes sense and take \( \mathbb{R} \) as the codomain.

**E.g:** Let \( f(x) = \frac{x^2}{x-1} \). Then we take the domain as \( \{ x \in \mathbb{R} \mid x \neq 1 \} \), the codomain as \( \mathbb{R} \).

**Definition:** Let \( f, g \) be two functions. We say they are equal, denoted by \( f = g \), if they have the same domain and codomain, and they have the same value at each point: \( f(x) = g(x) \).

**Definition:** Let \( f : X \to Y \) be a function and \( A \) be a subset of \( X \). Then we can define a new function \( g : A \to Y \) by \( g(a) = f(a) \) for all \( a \in A \). This function \( g \) is called the restriction of \( f \) to \( A \).

**Notation:** \( g = f|A \).

**E.g:** Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x \). Let \( h : \mathbb{R} \to \mathbb{R} \) be defined by \( h(x) = |x| \). Write \( \mathbb{R}^{\geq 0} \) for the set of nonnegative real numbers. Then

\[
f|\mathbb{R}^{\geq 0} = h|\mathbb{R}^{\geq 0}
\]

**Definition:** Let \( f : X \to Y \) be a function. The image of \( f \), denoted by \( \text{Im} f \) is defined by

\[
\text{Im} f = \{ f(x) \mid x \in X \}.
\]

**Remark:** We always have \( \text{Im} f \subset Y \). When \( \text{Im} f = Y \), we say \( f \) is surjective.

**E.g:** Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = \sin x \). Then \( \text{Im} f = [-1, 1] \).

**Definition:** Let \( f : X \to Y \) be a function. The graph of \( f \), denoted by \( G_f \) is defined by

\[
G_f = \{ (x, y) \in X \times Y \mid y = f(x) \} = \{ (x, f(x)) \mid x \in X \}.
\]

## 2 Composition of functions

**Definition:** Given two functions \( f : X \to Y \) and \( g : Y \to Z \), the composite of \( f \) and \( g \), denoted by \( g \circ f : X \to Z \), is defined by

\[
g \circ f(x) = g(f(x)) \text{ for all } x \in X.
\]
E.g: Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^4$, $g : \mathbb{R} \to \mathbb{R}$ be $g(x) = x + 1$. Then

$$f \circ g(x) = (x + 1)^4; g \circ f(x) = x^4 + 1; f \circ f(x) = x^{16}, g \circ g(x) = x + 2.$$ 

**Proposition:** Let $f : X \to Y, g : Y \to Z$ and $h : Z \to W$ be functions. Then

$$(h \circ g) \circ f = h \circ (g \circ f).$$