Solutions for Problems 1, Ex. 4, 5, 7.

**Problems I, Ex. 4 (p. 53)**

Prove the following statement concerning positive integers \( a, b, \) and \( c. \)

(i) \((a \text{ divides } b) \text{ and } (a \text{ divides } c) \Rightarrow a \text{ divides } (b+c).\)

(ii) \((a \text{ divides } b) \text{ or } (a \text{ divides } c) \Rightarrow a \text{ divides } (bc).\)

Let \( a, b, \) and \( c \) be positive integers.

(i) **Proof.** Suppose that \( a \) divides \( b \) and \( a \) divides \( c. \) Then by the definition of divides, we have \( b = aq_1 \) and \( c = aq_2 \) for some integers \( q_1 \) and \( q_2. \) Notice that \( b + c = a(q_1 + q_2), \) so \( b + c = a(q_1 + q_2) \) by distributivity. The sum of two integers is an integer, so \( q_1 + q_2 \) is an integer. Therefore \( a \) divides \( b + c \) by the definition of divides. ■

(ii) **Proof.** Suppose that \( a \) divides \( b \) or \( a \) divides \( c. \) Then by the definition of divides, we have \( b = aq_1 \) or \( c = aq_2 \) for some integers \( q_1 \) and \( q_2. \) If \( b = aq_1, \) then \( bc = a(cq_1), \) so \( a \) divides \( bc. \) Otherwise, if \( c = aq_2, \) then \( bc = a(bq_2), \) so \( a \) divides \( bc. \) Therefore, \( a \) divides \( bc \) because it is true in all (i.e., both) cases. ■
Problems 1, Ex. 5 (p. 53)

Which of the following conditions are necessary for the positive integer \( n \) to be divisible by 6 (proofs not necessary)?

(i) 3 divides \( n \).
(ii) 9 divides \( n \).
(iii) 12 divides \( n \).
(iv) \( n = 12 \).
(v) 6 divides \( n^2 \).
(vi) 2 divides \( n \) and 3 divides \( n \).

Recall that “\( P \) is necessary for \( Q \)” means “\( Q \) implies \( P \)”. So this exercise is asking, “Given that 6 divides \( n \), which of the following conditions are true?”

The answer is conditions (i), (v), and (vi). Proofs are not required, but here they are anyway:

(i) Suppose that 6 divides \( n \). Then \( n = 6q \) for some integer \( q \), so \( n = 3(2q) \). Therefore 3 divides \( n \).

(v) Suppose that 6 divides \( n \). Then \( n = 6q \) for some integer \( q \), so \( n^2 = 6(6q^2) \). Therefore 6 divides \( n^2 \).

(vi) Suppose that 6 divides \( n \). Then \( n = 6q \) for some integer \( q \), so \( n = 2(3q) \) and \( n = 3(2q) \). Therefore 2 divides \( n \) and 3 divides \( n \).
Problems I, Ex. 7 (p. 53)

Prove by contradiction the following statement concerning an integer $n$: $n^2$ is even $\Rightarrow n$ is even. [You may suppose that an integer $n$ is odd if and only if $n = 2q + 1$ for some integer $q$. This is proved later as Proposition 11.3.4.]

Proof. Let $n$ be an integer, and suppose that $n^2$ is even. Assume for contradiction that $n$ is odd. Then $n = 2q + 1$ for some integer $q$, so

$$n^2 = (2q + 1)^2$$
$$= 4q^2 + 4q + 1$$
$$= 2(2q^2 + 2q) + 1,$$

where the last two equalities are by distributivity. Hence $n^2$ is odd, which contradicts $n^2$ being even. Therefore our assumption was false; instead, $n$ must be even. $\blacksquare$