Solutions for Problems III, Ex. 1, 4, 14, 20.

**Problems III, Ex. 1 (p. 182)**

Of the 182 students who are taking three first year core Mathematics modules (Reasoning, Algebra and Calculus), 129 like Reasoning, 129 like Algebra, 129 like Calculus, 85 like Reasoning and Algebra, 89 like Reasoning and Calculus, 86 like Algebra and Calculus, and 54 like all three modules. How many of the students like none of the core modules?

1 student likes none of the core modules.

Let $R, A, C$ be the sets of students who like Reasoning, Algebra, and Calculus respectively. Now let $N$ be the students who like none of core modules. Since there are 182 students, we know that

$$|R \cup A \cup C| + |N| = 182$$

so by the inclusion–exclusion principle and by substitution, we find that

$$|R| + |A| + |C| - |R \cap A| - |R \cap C| - |A \cap C| + |R \cap A \cap C| + |N| = 182$$

$$129 + 129 + 129 - 85 - 89 - 86 + 54 + |N| = 182$$

$$|N| = 1.$$
Problems III, Ex. 4 (p. 182–183)

Prove (by induction on \( n \)) the general inclusion–exclusion principle which may be stated as follows.

Let \( A_1, A_2, \ldots, A_n \) be finite sets. For \( I = \{i_1, i_2, \ldots, i_r\} \subseteq \mathbb{N}_n \), write

\[
A_I = \bigcap_{i \in I} A_i = A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}.
\]

Then

\[
\left| \bigcup_{i=1}^n A_i \right| = \sum_{\varnothing \neq I \subseteq \mathbb{N}_n} (-1)^{|I|-1} |A_I|,
\]

summing over all non-empty subsets of \( \mathbb{N}_n \).

**Proof.** Proof by induction on \( n \).

*Base case:* If \( n = 1 \), then

\[
\left| \bigcup_{i=1}^1 A_i \right| = |A_1| = \sum_{\varnothing \neq I \subseteq \mathbb{N}_n} (-1)^{|I|-1} |A_I|.
\]

*Inductive step:* Suppose that

\[
\left| \bigcup_{i=1}^k X_i \right| = \sum_{\varnothing \neq I \subseteq \mathbb{N}_k} (-1)^{|I|-1} |X_I|
\]

for all collections of finite sets \( X_1, X_2, \ldots, X_k \). Now let \( A_1, A_2, \ldots, A_{k+1} \) be finite sets. Then

\[
\left| \bigcup_{i=1}^{k+1} A_i \right| = \left| \left( \bigcup_{i=1}^k A_i \right) \cup A_{k+1} \right|,
\]

so by the inclusion–exclusion principle for two sets,

\[
\left| \bigcup_{i=1}^{k+1} A_i \right| = \left| \bigcup_{i=1}^k A_i \right| + |A_{k+1}| - \left| \left( \bigcup_{i=1}^k A_i \right) \cap A_{k+1} \right|
\]

\[
= \left| \bigcup_{i=1}^k A_i \right| + |A_{k+1}| - \left| \bigcup_{i=1}^k (A_i \cap A_{k+1}) \right|.
\]

Now apply the inductive hypothesis on the collections of finite sets \( A_1, A_2, \ldots, A_k \) and \( A_1 \cap A_{k+1}, A_2 \cap A_{k+1}, \ldots, A_k \cap A_{k+1} \) to conclude that

\[
\left| \bigcup_{i=1}^{k+1} A_i \right| = \left( \sum_{\varnothing \neq I \subseteq \mathbb{N}_k} (-1)^{|I|-1} |A_I| \right) + |A_{k+1}| - \left( \sum_{\varnothing \neq I \subseteq \mathbb{N}_k} (-1)^{|I|-1} |A_I \cap A_{k+1}| \right)
\]

\[
= \sum_{\varnothing \neq I \subseteq \mathbb{N}_{k+1}} (-1)^{|I|-1} |A_I|.
\]

\( \blacksquare \)
Problems III, Ex. 14 (p. 184)

For \( n \in \mathbb{Z}^+ \), suppose that \( A \subseteq N_{2n} \) and \(|A| = n + 1\). Prove that \( A \) contains a pair of distinct integers \( a, b \) such that \( a \) divides \( b \).

[Let \( f(a) \) be the greatest odd integer which divides \( a \) and apply the pigeonhole principle to \( f \).]

Proof. Let \( n \in \mathbb{Z}^+ \) and suppose that \( A \subseteq N_{2n} \) with \(|A| = n + 1\). Notice that each positive integer \( c \) can be uniquely written as \( c = 2^i d \), where \( i \) is a non-negative integer and \( d \) is a positive odd integer (\( d \) is the greatest odd integer that divides \( c \)). Now let \( D = \{1, 3, \ldots, 2n - 1\} \) and define the function \( f : A \to D \) by \( f(c) = d \) where \( d \) is defined as above for each \( c \). Observe that \(|D| = n\), so by the pigeonhole principle, there are \( a, b \in A \) such that \( a < b \) and \( f(a) = f(b) \). By the definition of \( f \), we have \( a = f(a)2^j \) and \( b = f(b)2^k = f(a)2^k \) for some positive integers \( j < k \), so \( b = a(2^k - j) \). Therefore \( a \) divides \( b \). ■

Problems III, Ex. 20 (p. 185)

Use the pigeonhole principle to prove that, given ten distinct positive integers less than 107, there exist two disjoint subsets with the same sum.

Proof. Let \( A \) be a set of ten positive integers less than 107. Notice that the sum of any subset of \( A \) is at least 0 and at most \( 97 + 98 + \ldots + 106 = 10 \cdot (97 + 106)/2 = 1015 \). Denote the power set of \( A \) by \( \mathcal{P}(A) \) and define the function \( f : \mathcal{P}(A) \to \{0, 1, \ldots, 1015\} \) such that \( f(B) \) is the sum of elements in the subset \( B \). By the pigeonhole principle, since \(|\mathcal{P}(A)| = 1024 > 1016 = |\{0, 1, \ldots, 1015\}|\), there are distinct subsets \( C, D \in \mathcal{P}(A) \) such that \( f(C) = f(D) \), meaning that \( C \) and \( D \) have the same sum. Now note that \( C - D \) and \( D - C \) are disjoint, and observe that

\[
\begin{align*}
f(C - D) &= f(C) - f(C \cap D) \\
&= f(D) - f(C \cap D) \\
&= f(D - C).
\end{align*}
\]

Therefore \( C - D \) and \( D - C \) are disjoint subsets with the same sum. ■