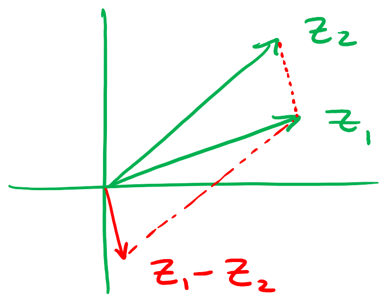
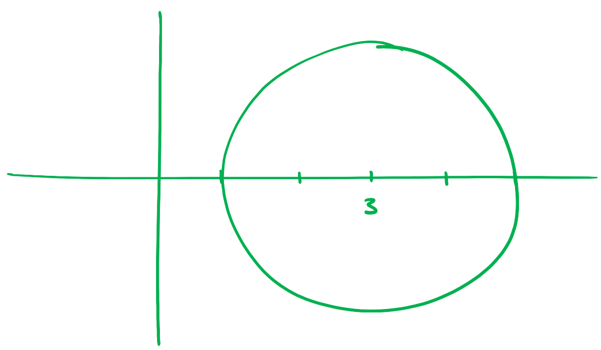
2. Regions in the Complex Plane

<u>Distance:</u> Let $z_1, z_2 \in \mathbb{C}$,

$|z_1 - z_2|$ = distance between z_1 and z_2.



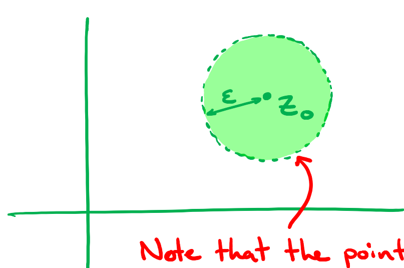
<u>Example:</u> Describe the set $\{z \in \mathbb{C} : |z-3| = 2\}$.



circle of radius 2 centered at 3.

Let $z_0 \in \mathbb{C}$, $\varepsilon > 0$.

<u>Def^n:</u> The set $\{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ is called a <u>neighborhood</u> or an <u>ε-neighbourhood</u> of z_0.



Note that the points with $|z - z_0| = \varepsilon$ are not included.

The inside of a circle, radius ε, centered at z_0.

Interior points & open sets

Let S be a set of complex numbers.

Let $z_0 \in S$.

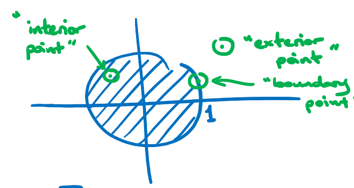
Def^n: z_0 is an <u>interior point</u> of S

$$\Updownarrow$$

S contains a neighborhood of z_0.

Examples:

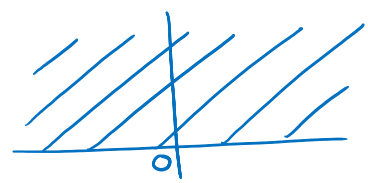
① $S = \{z \in \mathbb{C} : |z| \leq 1\}$



→ interior points of S: $\{z \in \mathbb{C} : |z| < 1\}$

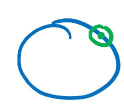
② $S = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ ← "upper half plane"

→ any point of S is an interior point.



③ $S = \{z \in \mathbb{C} : |z| = 1\}$ unit circle

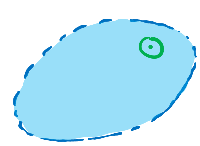
→ S has no interior points.



Def^n: $S \subseteq \mathbb{C}$ is <u>open</u>

$$\Updownarrow$$

every point in S is an interior point.



Examples: short for $\{z \in \mathbb{C} : |z|<1\}$

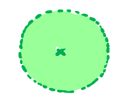
① $\{|z|<1\}$ is open.

② $\{\operatorname{Im} z > 0\}$ is open.

③ $\{|z| \le 1\}$ is not open.

④ $\{0 < |z| < 1\}$ is open.

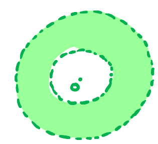
↳ "punctured disk"



⑤ $\{0 < |z| \le 1\}$ is not open.

⑥ $\{1 < |z| < 2\}$ is open.

↳ an <u>annulus</u>



⑦ $\{1 < |z| \le 2\}$ is not open.

<u>Boundaries & closed sets</u>

Given $S \subseteq \mathbb{C}$, let $S^c = \mathbb{C} - S$ denote the <u>complement</u> of S.

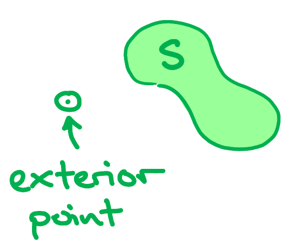
(Recall $S^c = \{z \in \mathbb{C} : z \notin S\}$.)

→ If $S = \{|z|<1\}$, $S^c = \{|z| \ge 1\}$.

<u>Def</u>^{n}: If $S \subseteq \mathbb{C}$, $z_0 \notin S$, we say z_0 is <u>exterior</u> to S

$\Updownarrow$

z_0 is an interior point of S^c.



Examples:

① $S = \{|z| < 1\}$

$\rightarrow$ exterior points of S: $\{|z| > 1\}$.

② $S = \{|z| = 1\}$

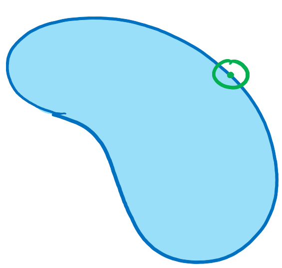
$\rightarrow$ exterior points of S: $\{|z| < 1\} \cup \{|z| > 1\}$.

$\uparrow$ set union

③ $S = \{|z| \geq 1\}$

$\rightarrow$ exterior points: $\{|z| < 1\}$.

Def^n: Given $S \subseteq \mathbb{C}$, $z_0 \in \mathbb{C}$,



z_0 is a boundary point of S

$\Updownarrow$

z_0 is not interior & not exterior to S.

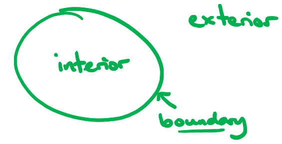
Equivalently, z_0 is a boundary point of S if every ε-neighborhood of z_0 contains points inside of S and outside of S.

Notation: $bS = \{$boundary points of $S\}$

("∂S" is another common notation for the boundary.)

Examples:

① $S = \{ |z| < 1 \}$



$\rightarrow$ boundary of S: $\{ |z| = 1 \}$

② $S = \{ |z| = 1 \}$

$\rightarrow$ boundary $bS = \{ |z| = 1 \}$.

③ $S = \{ \operatorname{Im} z > 0 \}$

$\rightarrow$ boundary $bS = \{ \operatorname{Im} z = 0 \} = \mathbb{R} \subset \mathbb{C}$.

④ $S = \mathbb{C}$

$\rightarrow$ S has no boundary points (S is open)

boundary $bS = \phi$. $\leftarrow$ empty set

⑤ $S = \{ 0 < |z| < 1 \}$

$\rightarrow$ $bS = \{0\} \cup \{ |z| = 1 \}$.

Def^n: $S \subseteq \mathbb{C}$ is closed

$\Updownarrow$

S contains its boundary (i.e. $bS \subseteq S$).

Remark $\rightarrow$ $\|$ $\Updownarrow$

S^c is open.

Examples:

1. $S = \{|z| \leq 1\}$ is closed.
2. $S = \{|z| = 1\}$ is closed.
3. $S = \{0 < |z| \leq 1\}$ is not closed.
4. $S = \{|z| < 1\}$ is not closed.
5. $S = \{\operatorname{Im} z > 0\}$ is not closed.
6. $S = \mathbb{C}$ is closed (and open).

the only subsets of $\mathbb{C}$ which are closed and open ("clopen") are $\mathbb{C}$ and $\emptyset$.

Def^n: The closure of a set $S \subseteq \mathbb{C}$ is the set

the notation $\bar{S}$ for closure is more common, but this could be confused with complex conjugation.

$$\operatorname{cl} S \overset{\text{def}}{=} \underbrace{S \cup bS}_{S \text{ together with its boundary}}.$$

Note: the closure of a set is closed.

Example: the closure of $\{0 < |z| < 1\}$ is $\{|z| \leq 1\}$.

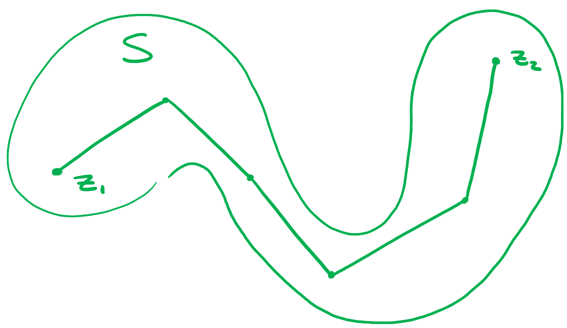
replace $<$ with $\leq$

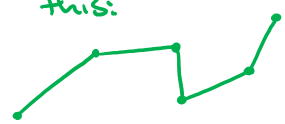
$0 \leq |z| \leq 1$

Connected sets, domains & regions

Def^n: An open set $S \subseteq \mathbb{C}$ is connected
$\Updownarrow$
any two points z_1, z_2 in S can be joined by a polygonal line contained in S.

something like this:



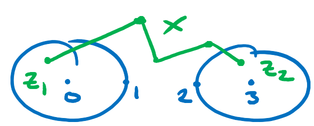


Examples:

① $\{|z| < 1\}$ is connected.

② $\{1 < |z| < 2\}$ is connected.

③ $\{|z|<1\} \cup \{|z-3|<1\}$ is not connected.



Def^n: A nonempty set $S \subseteq \mathbb{C}$ is called

these are the kinds of "nice" sets that we will usually work with

(i) a domain $\Longleftrightarrow$ S is open & connected;

(ii) a region $\Longleftrightarrow$ S is a domain together with some of its boundary points.

could be all or none.

Examples:

1. $\{|z| < 1\}$ is a domain.
2. $\{|z| \le 1\}$ is <u>not</u> a domain, but is a region.
3. $\{0 < |z| < 1\}$ is a domain.
4. $\{0 < |z| \le 1\}$ is <u>not</u> a domain, but is a region.
5. $\{|z| = 1\}$ is <u>not</u> a domain, and is <u>not</u> a region.

<u>Def^n</u>: $S \subseteq \mathbb{C}$ is <u>bounded</u>

$$\Updownarrow$$

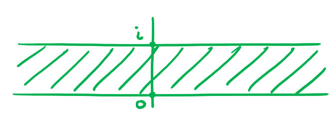
$$S \subseteq \{z \in \mathbb{C} : |z| < R\} \text{ for some } R > 0.$$

(That is, $S \subseteq \mathbb{C}$ is bounded if you can draw a large circle around the set.)

Examples:

1. $\{|z| < 100\}$ is bounded.
2. $\{1 < |z| < 2\}$ is bounded.
3. $\mathbb{C}$ is unbounded.
4. $\{\operatorname{Im} z > 0\}$ is unbounded.
5. $\{0 \le \operatorname{Im} z \le 1\}$ is unbounded.

   ← infinite <u>strip</u>



Let $z_0 \in \mathbb{C}$, $\varepsilon > 0$.

Def^n: the set $\{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$ is called a *deleted neighborhood* or *deleted ε-neighborhood* of z_0.

It is the usual ε-nbhd, with z_0 deleted.

Def^n: Given $S \subseteq \mathbb{C}$, $z_0 \in \mathbb{C}$,

z_0 is an *accumulation point* of S

$\Updownarrow$

every deleted neighborhood of z_0 contains some point(s) of S.

this says that there are points in S, distinct from z_0, which are arbitrarily close to z_0.

Examples:

① $S = \{0 < |z| < 1\}$

$\rightarrow$ accumulation points of S: $\{|z| \le 1\}$.

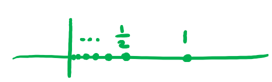
② $S = \{|z| \le 1\}$

$\rightarrow$ accumulation points of S: $\{|z| \le 1\}$.

③ $S = \mathbb{Z} \subset \mathbb{C}$

$\rightarrow$ no accumulation points.

④ $S = \{\frac{1}{n} : n \in \mathbb{Z}, n > 0\}$



$\rightarrow$ the only accumulation point is 0.

0 is an accumulation point of S since for any $\varepsilon > 0$, there is $n \in \mathbb{Z}$, $n > 0$, large enough so that $\frac{1}{n} \in \{0 < |z| < \varepsilon\}$.