3. Functions of a Complex Variable

Let $S \subseteq \mathbb{C}$.

A <u>function</u> $f: S \to \mathbb{C}$ is a rule

$$S \ni z \mapsto w = f(z) \in \mathbb{C}.$$

$\rightsquigarrow$ We will call f a <u>function defined on S.</u>

(f will be assumed to take values in $\mathbb{C}$.)

Convention:
$$z = x + iy$$
$$w = u + iv$$

$\rightsquigarrow$ we write $\underline{f(z) = u(x,y) + iv(x,y)} = w$

we can think of $f(z)$ as being two real functions of two real variables (x & y).

$$u(x,y) = \operatorname{Re} f(z), \quad v(x,y) = \operatorname{Im} f(z)$$

Examples:

① $f(z) = z^2 \qquad (S = \mathbb{C})$

$$z^2 = (x+iy)^2 = x^2 - 2ixy - y^2$$

$\rightsquigarrow \quad u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy.$

② $f(z) = \frac{1}{z}$ $\quad (S = \{z \neq 0\})$

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$\rightsquigarrow \quad u(x,y) = \frac{x}{x^2+y^2}, \quad v(x,y) = \frac{-y}{x^2+y^2}$$

Sometimes it is convenient to use polar coordinates and write (for $z \neq 0$)

$$f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta).$$

Example: $f(z) = \frac{1}{z}$, $z \neq 0$

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

use the conjugate of Euler's formula

$$\rightsquigarrow \quad u(r,\theta) = \frac{\cos\theta}{r}, \quad v(r,\theta) = \frac{-\sin\theta}{r}.$$

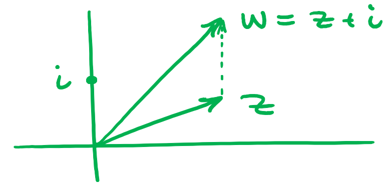
Functions as transformations

Sometimes it is helpful to view a particular function as a transformation of the complex plane.

Examples:

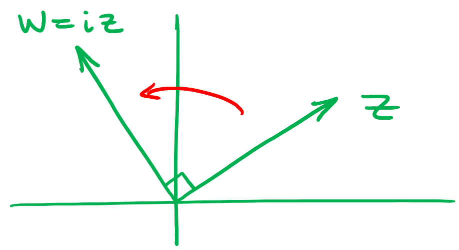
① $f(z) = z + i$

$\rightsquigarrow$ translation by i



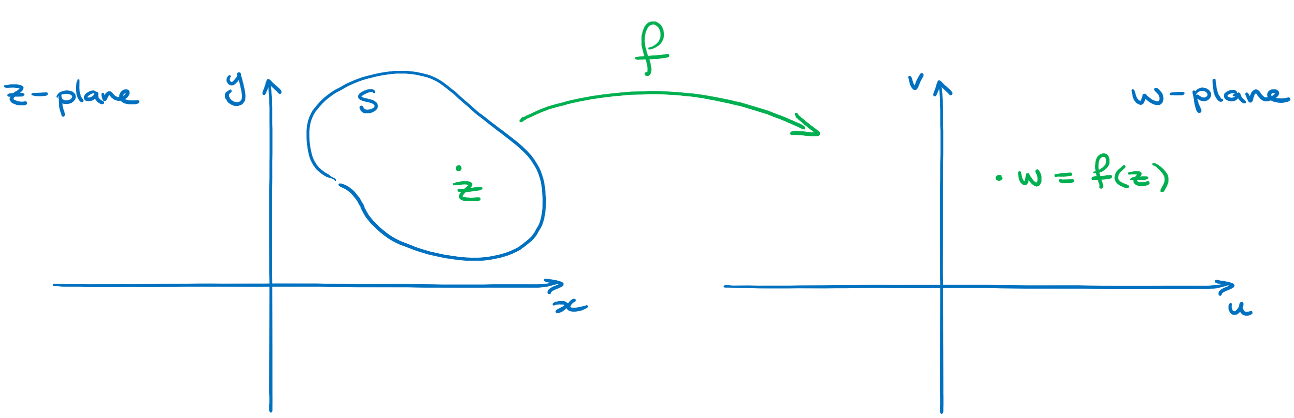
② $f(z) = iz$

$\rightsquigarrow$ rotation by $\frac{\pi}{2}$ (counter-clockwise)



Functions as mappings

Frequently we think of complex functions as <u>mappings</u> from (part of) the "z-plane" to (part of) the "w-plane".



Example: $f(z) = z^2$

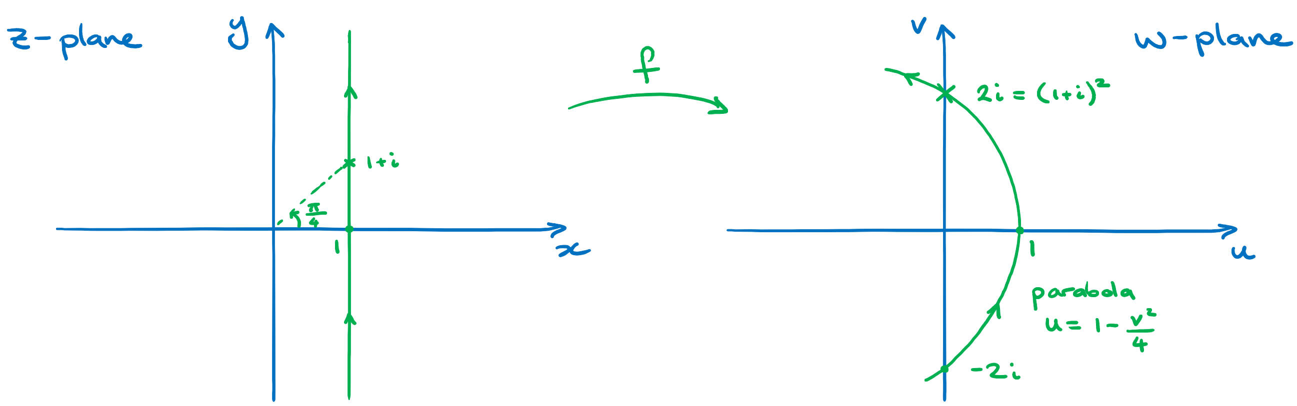
(a) Find the image under f of the line $\{\operatorname{Re} z = 1\}$.

$\rightsquigarrow$ $\{\operatorname{Re} z = 1\} = \{1 + it : t \in \mathbb{R}\}$

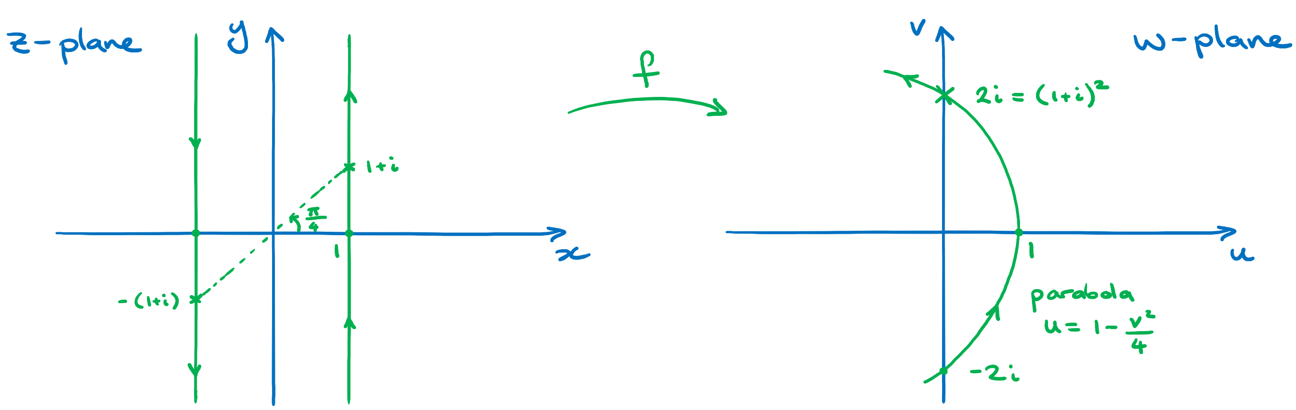
$f(1+it) = (1+it)^2 = 1 + 2it - t^2 = u(t) + iv(t)$

$\rightsquigarrow$ $\underline{u(t) = 1 - t^2, \quad v(t) = 2t}$

parametrizes the curve $u = 1 - \frac{v^2}{4}$, a sideways parabola



Remark: Since $(-z)^2 = z^2$, the line $\{\operatorname{Re} z = -1\}$ has exactly the same image:

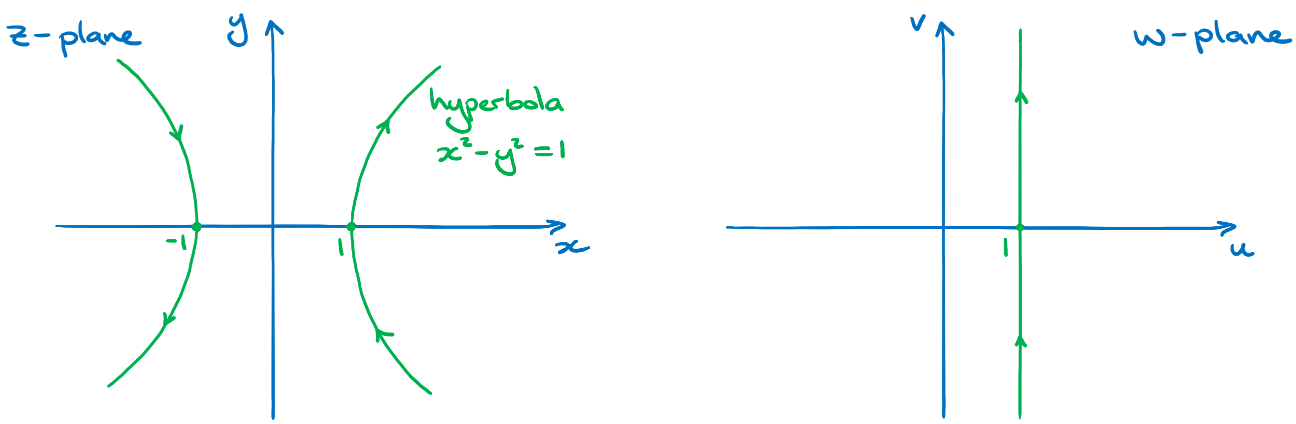


(b) Find the preimage (inverse image) under f of the line $\{\underbrace{\mathrm{Re}\, w}_{u} = 1\}$.

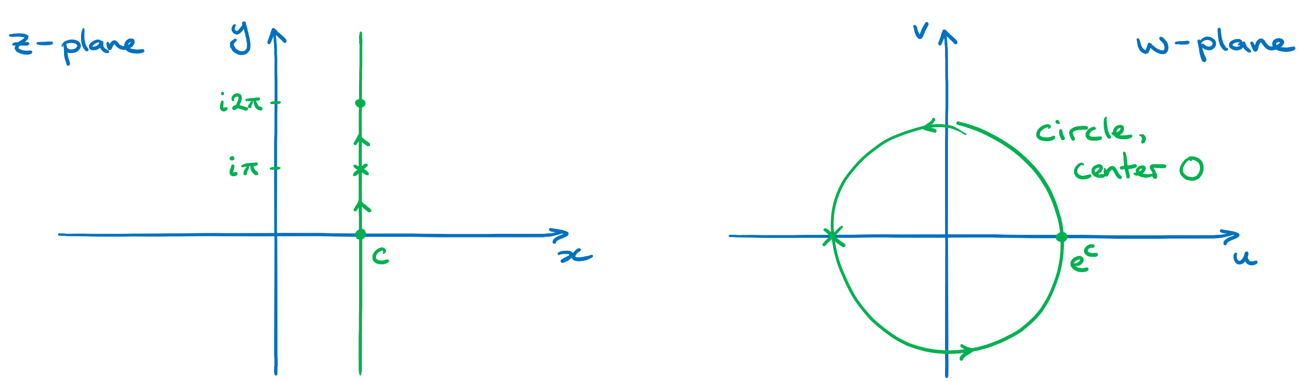
$$f(z) = z^2 = x^2 - y^2 + 2ixy,$$

$$u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy$$

$$\rightsquigarrow \quad \underbrace{u = x^2 - y^2 = 1}_{\text{unit hyperbola}}$$

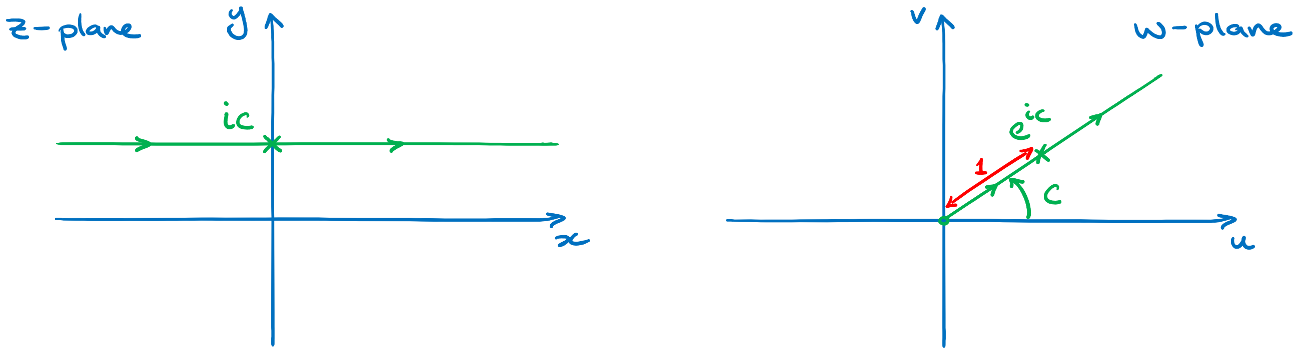


Example: $f(z) = e^z = e^x e^{iy}, \quad z = x + iy$



$$f(c + iy) = e^c \underline{\underline{e^{iy}}}$$

"phase"



$$f(x+ic) = \underline{\underline{e^x}}\, e^{ic}$$

$\longrightarrow$ recall $\lim_{x \to \infty} e^x = \infty$

$\lim_{x \to -\infty} e^x = 0.$