3. Functions of a Complex Variable

Let \( S \subseteq \mathbb{C} \).

A function \( f: S \rightarrow \mathbb{C} \) is a rule
\[
S \ni z \mapsto w = f(z) \in \mathbb{C}.
\]

We will call \( f \) a function defined on \( S \).

(f will be assumed to take values in \( \mathbb{C} \).

Convention:
\[
zeq z = x + iy
\]
\[
w = u + iv
\]

\( \leadsto \) we write
\[
f(z) = u(x,y) + iv(x,y) = w
\]

we can think of \( f(z) \) as being two real functions of two real variables \((x, y)\).

\[
u(x,y) = \text{Re} \ f(z) , \quad v(x,y) = \text{Im} \ f(z)
\]

Examples:

1. \( f(z) = z^2 \) \((S = \mathbb{C})\)
\[
z^2 = (x + iy)^2 = x^2 - 2ixy - y^2
\]
\( \leadsto \)
\[
u(x,y) = x^2 - y^2 , \quad v(x,y) = 2xy.
\]
2. \( f(z) = \frac{1}{z} \quad (S = \{ z \neq 0 \}) \)

\[ \frac{1}{z} = \frac{1}{z + iy} = \frac{z - iy}{z^2 + y^2} \]

\[ \Rightarrow u(x, y) = \frac{x}{x^2 + y^2}, \quad v(x, y) = \frac{-y}{x^2 + y^2} \]

Sometimes it is convenient to use polar coordinates and write (for \( z \neq 0 \))

\[ f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta) \]

Example: \( f(z) = \frac{1}{z}, \quad z \neq 0 \)

\[ \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r} \left( \cos \theta - i \sin \theta \right) \]

\[ \text{use the conjugate of Euler's formula} \]

\[ \Rightarrow u(r, \theta) = \frac{\cos \theta}{r}, \quad v(r, \theta) = \frac{-\sin \theta}{r} \]

Functions as transformations

Sometimes it is helpful to view a particular function as a transformation of the complex plane.
Examples:

1. \( f(z) = z + i \)
   \( \Rightarrow \) translation by \( i \)

2. \( f(z) = iz \)
   \( \Rightarrow \) rotation by \( \frac{\pi}{2} \) (counter-clockwise)

Functions as mappings

Frequently we think of complex functions as mappings from (part of) the “\( z \)-plane” to (part of) the “\( w \)-plane.”
Example: \( f(z) = z^2 \)

(a) Find the image under \( f \) of the line \( \Re z = 13 \).

\[ \Rightarrow \Re z = 13 = \Re (1+it) : t \in \mathbb{R} \]

\[ f(1+it) = (1+it)^2 = 1 + 2it - t^2 = u(t) + iv(t) \]

\[ \Rightarrow u(t) = 1 - t^2, \quad v(t) = 2t \]

\[ \overset{\text{parametrizes the curve}}{\Rightarrow} \quad u = 1 - \frac{v^2}{4}, \text{ a sideways parabola} \]

**Remark:** Since \((-z)^2 = z^2\), the line \( \Re z = -13 \) has exactly the same image.
(b) Find the preimage (inverse image) under $f$ of the line $\text{Re}(w) = 13$.

\[ f(z) = z^2 = x^2 - y^2 + 2ixy, \]
\[ u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy \]
\[ \Rightarrow u = x^2 - y^2 = 1 \]
\[ \text{unit hyperbola} \]

Example: $f(z) = e^z = e^x e^{iy}$, $z = x + iy$

\[ f(c + iy) = e^c e^{iy} \]

\[ \text{Circle, center O} \]

\[ \text{Phase} \]
\[ f(x + ic) = e^x e^{ic} \]

L \rightarrow \text{recall} \quad \lim_{x \to \infty} e^x = \infty \quad \lim_{x \to -\infty} e^x = 0. \]