5. Derivatives

Function $f: S \to \mathbb{C}$, $S \subseteq \mathbb{C}$,

z_0 an interior point of S.

Def^n: f is (complex) differentiable at z_0

$\Updownarrow$

$$f'(z_0) = \underbrace{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}_{\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}} \text{ exists.}$$

Notation for derivatives:

$\Delta z = z - z_0$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Often we drop the subscript on z_0 & write:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

or, letting $w = f(z)$ & $\Delta w = f(z + \Delta z) - f(z)$,

notation for f' $\longrightarrow$

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}.$$

Examples:

(1) $f(z) = z^2$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \frac{\cancel{z^2} + 2z\Delta z + (\Delta z)^2 - \cancel{z^2}}{\Delta z}$$

$$= 2z + \Delta z$$

$$\leadsto f'(z) = \lim_{\Delta z \to 0} [2z + \Delta z] = 2z.$$

(2) $f(z) = \bar{z} = x - iy$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\cancel{\bar{z}} + \overline{\Delta z} - \cancel{\bar{z}}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} \xrightarrow[\Delta z \to 0]{} ?$$

$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ does not exist:

$$\frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \begin{cases} 1 & \text{if } \Delta y = 0 \\ -1 & \text{if } \Delta x = 0 \\ \ldots \end{cases}$$

$\leadsto f(z) = \bar{z} = x - iy$ is <u>not</u> (complex) differentiable at any point in $\mathbb{C}$.

Writing $f(z) = \bar{z} = x - iy = u(x,y) + iv(x,y)$.
The functions $u(x,y) = x$ and $v(x,y) = -y$ are nice <u>real</u> differentiable functions on $\mathbb{R}^2$.
But this does not make f complex diff^{ble}.

(3) $f(z) = z\bar{z} = |z|^2 = x^2 + y^2$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{(z+\Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \underbrace{\bar{z} + \overline{\Delta z}}_{\substack{\downarrow \Delta z \to 0 \\ \bar{z}}} + \underbrace{z \frac{\overline{\Delta z}}{\Delta z}}_{\substack{\text{no limit as} \\ \Delta z \to 0 \text{ unless} \\ z = 0.}}$$

$\leadsto$ $f'(0) = 0$, but

$f'(z)$ does not exist for $z \neq 0$.

Again $u(x,y) = x^2$ and $v(x,y) = y^2$ are nice real differentiable functions.
We say that f is $\mathbb{R}$-differentiable, but it is not $\mathbb{C}$-differentiable (except at $z=0$).

$$\boxed{\begin{array}{c}\underline{\text{General Rule:}} \\ \bar{z}\text{'s are bad when it comes} \\ \text{to (complex) differentiability}\end{array}}$$

More on this later

$z^5 + 2z^2 + z + 1$ $\leftarrow$ $\mathbb{C}$-diff$^{\text{ble}}$

$$\left.\begin{array}{l} z^4\bar{z} + 2\bar{z}^2 + 3z \\ \bar{z}^3 + 2z^2 \end{array}\right\} \longleftarrow \text{not } \mathbb{C}\text{-diff}^{\text{ble}}$$

Rules of Differentiation

The usual rules apply, with proofs like in the real case (no need to prove these, but look in the book).

① $(f+g)'(z) = f'(z) + g'(z)$

② $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

③ $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - g'(z)f(z)}{(g(z))^2}$

④ $\frac{d}{dz}[c] = 0$, c a constant

$\frac{d}{dz}[z^n] = nz^{n-1}$

⑤ Chain rule:

$$\frac{d}{dz}\Big[f(g(z))\Big] = f'(g(z)) \cdot g'(z).$$

⇝ We can differentiate polynomials and rational functions in z.

$\frac{d}{dz}(z^3 - 5z^2 + 2z + i) = 3z^2 - 10z + 2$

$\frac{d}{dz}\left[\frac{1}{z^2+1}\right] = \frac{-2z}{(z^2+1)^2} \quad (z \neq \pm i)$

Theorem: If f is diff^{ble} at z_0, then f is continuous at z_0.

Proof: Easy.

We know $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.

For $z \neq z_0$,

$$f(z) - f(z_0) = \frac{f(z)-f(z_0)}{z-z_0} \cdot (z-z_0),$$

so

$$\lim_{z\to z_0} [f(z)-f(z_0)] = \underbrace{\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}}_{=f'(z_0)} \cdot \underbrace{\lim_{z\to z_0} (z-z_0)}_{=0}.$$

Hence $\lim_{z\to z_0} [f(z) - f(z_0)] = 0$,

i.e. $$\underbrace{\lim_{z\to z_0} f(z) = f(z_0)}_{f \text{ is continuous at } z_0!}$$

$\square$