5. Derivatives

Function \( f: S \rightarrow \mathbb{C}, \ S \subseteq \mathbb{C}, \)
\( z_0 \) an interior point of \( S. \)

Def.: \( f \) is (complex) differentiable at \( z_0 \)
\[
f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}
\]
exists.

\[
\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}
\]

Notation for derivatives:

\[
f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}
\]

Often we drop the subscript on \( z_0 \)
& write:

\[
f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}
\]

or, letting \( w = f(z) \) & \( \Delta w = f(z + \Delta z) - f(z), \)

notation for \( f' \) \( \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}. \)
Examples:

1. \( f(z) = z^2 \)

\[
\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = 2z + \Delta z
\]

\[\Rightarrow f'(z) = \lim_{\Delta z \to 0} [2z + \Delta z] = 2z. \]

2. \( f(z) = \overline{z} = x - iy \)

\[
\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{z+\Delta z} - \overline{z}}{\Delta z} = \frac{\Delta \overline{z}}{\Delta z} \rightarrow ?
\]

\[\lim_{\Delta z \to 0} \frac{\Delta \overline{z}}{\Delta z} \text{ does not exist:} \]

\[
\frac{\Delta \overline{z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \begin{cases} 
1 & \text{if } \Delta y = 0 \\
-1 & \text{if } \Delta x = 0
\end{cases}
\]

\[\Rightarrow f(z) = \overline{z} = x - iy \text{ is not (complex) differentiable at any point in } \mathbb{C}. \]

Writing \( f(z) = \overline{z} = x - iy = u(x,y) + iv(x,y) \).

The functions \( u(x,y) = x \) and \( v(x,y) = -y \) are nice real differentiable functions on \( \mathbb{R}^2 \). But this does not make \( f \) complex differentiable.
\[ f(z) = |z|^2 = x^2 + y^2 \]

\[ \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{(z + \Delta z)(\bar{z} + \Delta \bar{z}) - z\bar{z}}{\Delta z} \]

\[ = \frac{\bar{z} + \Delta \bar{z} + z\frac{\Delta z}{\bar{z}}}{\Delta z} \]

\[ \downarrow \Delta z \to 0 \]

\[ \frac{\Delta z}{\bar{z}} \text{ no limit as } \Delta z \to 0 \text{ unless } z = 0. \]

\[ \Rightarrow f'(z) = 0, \text{ but } f'(z) \text{ does not exist for } z \neq 0. \]

Again \( u(x,y) = x^2 \) and \( v(x,y) = y^2 \) are nice \underline{real} differentiable functions.

We say that \( f \) is \underline{R}-differentiable, but it is not \underline{C}-differentiable (except at \( z = 0 \)).

**General Rule:**

\underline{\bar{z}}'s \underline{are bad when it comes to (complex) differentiability}

More on this later:

- \( z^5 + 2z^2 + z + 1 \) \( \in \) \underline{C-diff}\*
- \( z^4 \bar{z} + 2\bar{z}^2 + 3z \) \( \notin \) \underline{C-diff}\*
- \( \bar{z}^3 + 2z^2 \)
Rules of Differentiation

The usual rules apply, with proofs like in the real case (no need to prove these, but look in the book).

1. \((f + g)'(z) = f'(z) + g'(z)\)

2. \(((fg)'(z) = f'(z)g(z) + f(z)g'(z)\)

3. \((\frac{f}{g})'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}\)

4. \(\frac{d}{dz} [c] = 0, \quad c \text{ a constant}\)

5. \(\frac{d}{dz} [z^n] = nz^{n-1}\)

5. Chain rule:

\[
\frac{d}{dz} [f(g(z))] = f'(g(z)) \cdot g'(z).
\]

\(\Rightarrow\) We can differentiate polynomials and rational functions in \(z\).

\[
\frac{d}{dz} (z^3 - 5z^2 + 2z + i) = 3z^2 - 10z + 2
\]

\[
\frac{d}{dz} \left[ \frac{1}{z^2 + 1} \right] = \frac{-2z}{(z^2 + 1)^2} \quad (z \neq \pm i)
\]
Theorem: If $f$ is differentiable at $z_0$, then $f$ is continuous at $z_0$.

Proof: Easy.

We know $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

For $z \neq z_0$,

$$f(z) - f(z_0) = \frac{f(z) - f(z_0)}{z - z_0} \cdot (z - z_0),$$

so

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0) = \frac{f'(z_0)}{1} = 0.$$

Hence $\lim_{z \to z_0} [f(z) - f(z_0)] = 0$,

i.e.

$$\lim_{z \to z_0} f(z) = f(z_0)$$

$f$ is continuous at $z_0$!