6. The Cauchy-Riemann Equations

We have seen that $u(x,y)$ & $v(x,y)$ being (real) differentiable is not enough for $f(z) = u(x,y) + i\,v(x,y)$ to be (complex) differentiable.

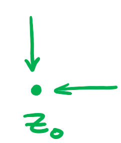
$\rightsquigarrow$ How do we interpret the complex differentiabivity of f in terms of u & v?

Fix $S \subseteq \mathbb{C}$, $f : S \to \mathbb{C}$, $z_0 \in S$ interior.

Assume $f'(z_0)$ exists.

Write $f(z) = u(x,y) + iv(x,y)$ and consider two ways of calculating

$$f'(z_0) = \lim_{\Delta z \to 0} \underbrace{\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}}_{\frac{\Delta w}{\Delta z}}$$



which we already know exists.

Write $\Delta w = \Delta u + i \Delta v$

& $\Delta z = \Delta x + i \Delta y$

$$\rightsquigarrow f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u}{\Delta z} + i \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta v}{\Delta z}.$$

If $\Delta z = \Delta x \quad (\Delta y = 0)$

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i\,\frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}.$$

Letting $\Delta x \to 0$, $\Delta z \to 0$ and we get

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\,\frac{\partial v}{\partial x}(x_0,y_0)$$

1 $$\boxed{f'(z_0) = u_x(x_0,y_0) + i\,v_x(x_0,y_0)}$$

Now if $\Delta z = i\Delta y \quad (\Delta x = 0)$

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0,y_0+\Delta y) - u(x_0,y_0)}{\underbrace{i\Delta y}_{\Delta z}} + \cancel{i}\,\frac{v(x_0,y_0+\Delta y) - v(x_0,y_0)}{\cancel{i}\Delta y}$$

$$\rightsquigarrow \frac{\Delta w}{\Delta z} = \frac{v(x_0,y_0+\Delta y) - v(x_0,y_0)}{\Delta y} - i\,\frac{u(x_0,y_0+\Delta y) - u(x_0,y_0)}{\Delta y}.$$

Letting $\Delta y \to 0$, $\Delta z \to 0$ and we get

2 $$\boxed{f'(z_0) = v_y(x_0,y_0) - i\,u_y(x_0,y_0).}$$

Comparing 1 and 2 we have

$$\left.\begin{aligned} u_x(x_0,y_0) &= v_y(x_0,y_0) \\ u_y(x_0,y_0) &= -v_x(x_0,y_0). \end{aligned}\right\} \text{Cauchy-Riemann Equations}$$

⇝ **Theorem:** If $f'(z_0)$ exists then

$$\left.\begin{aligned} u_x(x_0,y_0) &= v_y(x_0,y_0) \\ u_y(x_0,y_0) &= -v_x(x_0,y_0) \end{aligned}\right\} \text{Cauchy-Riemann}$$

and

$$f'(z_0) = u_x(x_0,y_0) + i v_x(x_0,y_0).$$

What about the converse?

Theorem: $f(z) = u(x,y) + iv(x,y)$.

Suppose $u(x,y)$ & $v(x,y)$ are differentiable at (x_0,y_0).

If $\boxed{u_x = v_y \quad \& \quad u_y = -v_x}$ at (x_0,y_0) ← Cauchy-Riemann

then $f'(z_0)$ exists.

Note:

1. The proof is easy, but we skip it.
2. Recall that a sufficient condition for $u(x,y)$ to be differentiable at (x_0,y_0) is that the partial derivatives u_x and u_y exist and are continuous at (x_0,y_0).

Examples:

① $f(z) = z^2 = x^2 - y^2 + 2ixy$

$\rightsquigarrow\ u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy$

$u_x = 2x \qquad\qquad v_x = 2y$

$u_y = -2y \qquad\qquad v_y = 2x$

$\rightsquigarrow\ u_x = v_y, \qquad u_y = -v_x$ ✓

② $f(z) = e^z = e^x \cos y + i e^x \sin y$

$\left(e^z = e^x e^{iy} = e^x(\cos y + i \sin y)\right)$

$\rightsquigarrow\ u(x,y) = e^x \cos y, \quad v(x,y) = e^x \sin y$

$u_x = e^x \cos y \qquad\qquad v_x = e^x \sin y$

$u_y = -e^x \sin y \qquad\qquad v_y = e^x \cos y$

$\rightsquigarrow\ u_x = v_y, \quad u_y = -v_x$ ✓

Cauchy-Riemann equations hold

$\rightsquigarrow\ f(z) = e^z$ is $\mathbb{C}$-diff^{ble},

$f'(z) = u_x(x,y) + i v_x(x,y)$

$= e^x \cos y + i e^x \sin y = e^z$

$$\boxed{\frac{d}{dz}(e^z) = e^z}$$

③ $f(z) = \bar{z} = x - iy$

$\rightsquigarrow \quad u(x,y) = x, \quad v(x,y) = -y$

$u_x = 1 \qquad v_x = 0$

$u_y = 0 \qquad v_y = -1$

$\rightsquigarrow \quad u_x = 1 \neq v_y = -1$

$\rightsquigarrow$ f is <u>not</u> $\mathbb{C}$-diff^ble.

④ $f(z) = z\bar{z} = |z|^2 = x^2 + y^2$

$\rightsquigarrow \quad u(x,y) = x^2 + y^2, \quad \underline{v(x,y) = 0}$

f is real-valued

$u_x = 2x \qquad v_x = 0$

$u_y = 2y \qquad v_y = 0$

Cauchy-Riemann equations only hold at $(x,y) = (0,0)$, i.e. $z = 0$.

$\rightsquigarrow$ f is not $\mathbb{C}$-diff^ble, except at $z=0$.

<u>Summary:</u>

$f = u + iv$ is $\mathbb{C}$-diff^ble $\iff$ f is $\mathbb{R}$-diff^ble <u>and</u> $u_x = v_y$, $u_y = -v_x$.

Cauchy-Riemann eqns in polar coord.s

$$z = x + iy = re^{i\theta}, \quad z \neq 0$$

$$\rightsquigarrow \quad x = r\cos\theta, \quad y = r\sin\theta$$

$$f(z) = u(r,\theta) + i\,v(r,\theta)$$

$$\rightsquigarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x \cos\theta + u_y \sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

$$u_r = u_x \cos\theta + u_y \sin\theta$$

$$u_\theta = -u_x r\sin\theta + u_y r\cos\theta$$

similarly:

$$v_r = v_x \cos\theta + v_y \sin\theta$$

$$v_\theta = -v_x r\sin\theta + v_y r\cos\theta$$

Applying the Cauchy-Riemann equations $u_x = v_y$ & $u_y = -v_x$ we get:

$$\boxed{\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ v_r &= -\frac{1}{r} u_\theta \end{aligned}}$$

Cauchy-Riemann equations in polar coord.s

Theorem:

$f(z) = u(r,\theta) + iv(r,\theta)$,

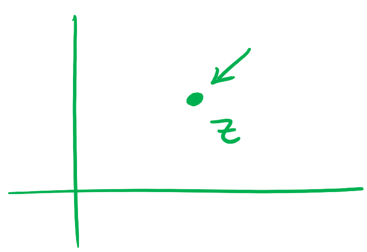
$u(r,\theta)$, $v(r,\theta)$ diff$^{\text{ble}}$, then

(i) $f'(z)$ exists $\Longleftrightarrow$ $u_r = \frac{1}{r} v_\theta$, $v_r = -\frac{1}{r} u_\theta$;

(ii) $f'(z) = e^{-i\theta}\big(u_r(r,\theta) + i v_r(r,\theta)\big)$.

Proof of (ii):

We know that $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists, so we may restrict to the case where Δz is radial in order to compute it.



$z = re^{i\theta}$

take $\Delta z = (\Delta r) e^{i\theta}$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{u(r+\Delta r,\theta) - u(r,\theta)}{e^{i\theta}\Delta r} + i\frac{v(r+\Delta r,\theta) - v(r,\theta)}{e^{i\theta}\Delta r}$$

$$= e^{-i\theta}\left[\frac{u(r+\Delta r,\theta) - u(r,\theta)}{\Delta r}\right] + ie^{-i\theta}\left[\frac{v(r+\Delta r,\theta) - v(r,\theta)}{\Delta r}\right]$$

$$\Big\downarrow \; \Delta r \to 0$$

$$f'(z) = e^{-i\theta}\left[u_r(r,\theta) + iv_r(r,\theta)\right].$$

$\square$

Examples:

① $f(z) = \frac{1}{z}$, $z \neq 0$.

(We know $f'(z) = \frac{-1}{z^2}$ by the quotient rule.)

$$f(z) = \frac{1}{re^{i\theta}} = \frac{e^{-i\theta}}{r}$$
$$= \frac{\cos\theta}{r} - i\,\frac{\sin\theta}{r}$$

$\rightsquigarrow$ $u(r,\theta) = \frac{\cos\theta}{r}$, $v(r,\theta) = \frac{-\sin\theta}{r}$

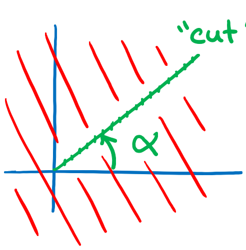
$\rightsquigarrow$ $u_r = \frac{-\cos\theta}{r^2}$, $v_r = \frac{\sin\theta}{r^2}$

$u_\theta = \frac{-\sin\theta}{r}$, $v_\theta = \frac{-\cos\theta}{r}$.

So $u_r = \frac{1}{r} v_\theta$, $v_r = -\frac{1}{r} u_\theta$ ✓

and

$$f'(z) = e^{-i\theta}(u_r + i v_r)$$
$$= e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + i\,\frac{\sin\theta}{r^2}\right)$$
$$= \frac{-1}{r^2} e^{-i\theta}(\cos\theta - i\sin\theta)$$
$$= -\frac{1}{r^2} e^{-2i\theta} = -\frac{1}{r^2 e^{2i\theta}} = -\frac{1}{z^2}.$$ ✓



② Fix an angle α, $-\pi < \alpha \leq \pi$.

Define $f(z) = \sqrt[3]{r}\, e^{i\theta/3}$ for $r > 0$, $\alpha < \theta < \alpha + 2\pi$.

$$\rightsquigarrow \quad f(z) = \underbrace{\sqrt[3]{r}\cos\left(\tfrac{\theta}{3}\right)}_{u(r,\theta)} + i\,\underbrace{\sqrt[3]{r}\sin\left(\tfrac{\theta}{3}\right)}_{v(r,\theta)}$$

$r u_r = \frac{1}{3}\sqrt[3]{r}\cos\left(\frac{\theta}{3}\right)$, $u_\theta = -\frac{1}{3}\sqrt[3]{r}\sin\left(\frac{\theta}{3}\right)$

$r v_r = \frac{1}{3}\sqrt[3]{r}\sin\left(\frac{\theta}{3}\right)$, $v_\theta = \frac{1}{3}\sqrt[3]{r}\cos\left(\frac{\theta}{3}\right)$

Cauchy-Riemann equations hold!