8. Harmonic Functions

$\Omega \subseteq \mathbb{R}^2$ open,

$u : \Omega \to \mathbb{R}$ twice diff^{ble}

Def^n: u is harmonic in Ω

$$\Updownarrow$$

"balancing" condition for u_{xx} & u_{yy}.

$$\underbrace{u_{xx}(x,y) + u_{yy}(x,y) = 0}_{\text{Laplace's equation!}} \quad \text{in } \Omega.$$

Theorem: If $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain $D \subseteq \mathbb{C}$, then $u(x,y)$ and $v(x,y)$ are harmonic in D.

Proof: We'll show in a later lecture that u & v have partial derivatives of any order (see sections 54–57 in the book), for now we will just assume that the second derivatives exist.

C.-R. eqns: $u_x = v_y$, $u_y = -v_x$

Recall $v_{xy} = v_{yx}$

$\rightsquigarrow$ $u_{xx} = v_{yx}$, $u_{yy} = -v_{xy}$

$\rightsquigarrow$ $\boxed{u_{xx} + u_{yy} = 0.}$

Similarly for v, $u_{xy} = v_{yy}$, $u_{yx} = -v_{xx}$

$$\rightsquigarrow \boxed{v_{xx} + v_{yy} = 0.}$$

$\square$

<u>Example:</u> $v(x,y) = e^x \sin y$

$v_{xx} = e^x \sin y$, $v_{yy} = -e^x \sin y$

$\rightsquigarrow$ $v_{xx} + v_{yy} = 0$ <u>v is harmonic</u>

of course!

e^z is <u>entire</u> and $e^z = e^x \cos y + i \boxed{e^x \sin y}$

<u>Def^n:</u> If u & v are harmonic in D then v is a <u>harmonic conjugate</u> of u

$\Updownarrow$

the C.R. eqns. $u_x = v_y$, $u_y = -v_x$ hold.

(This definition was made so that we would have the following theorem:)

<u>Theorem:</u> v is a harmonic conjugate of u

$\Updownarrow$

$f(z) = u(x,y) + iv(x,y)$ is analytic.

<u>Examples:</u>

① $e^x \sin y$ is a harmonic conjugate of $e^x \cos y$. (See above.)

② $u(x,y) = 2xy$ is a harmonic function.

$u_{xx} = u_{yy} = 0$

so $u_{xx} + u_{yy} = 0$

Find its harmonic conjugates in $\mathbb{R}^2$,

that is, find all $v(x,y)$ such that

$u_x = v_y$ and $u_y = -v_x$.

$\rightsquigarrow$ solve $v_x = -2x$, $v_y = 2y$

$$\downarrow$$

$$v(x,y) = -x^2 + \varphi(y)$$

$$v_y = \varphi'(y) = 2y \rightsquigarrow \varphi = y^2 + C.$$

So $\boxed{v(x,y) = y^2 - x^2 + C.}$

Note: $$\begin{aligned} u(x,y) + iv(x,y) &= 2xy + i(y^2 - x^2) + iC \\ &= -i(x+iy)^2 + iC \\ &= -iz^2 + iC \end{aligned}$$

Remark:

If v is a harmonic conjugate of u

then $-u$ is a harmonic conjugate of v.

(If $u+iv$ is analytic then so is

$-i(u+iv) = \underline{v + i(-u)}$.)

<u>Question</u> (for later):

Does every harmonic function in two variables arise as the real part of an analytic function?

→ <u>equivalently</u>:

Does every harmonic function in two variables have a harmonic conjugate?