8. Harmonic Functions

$\Omega \subseteq \mathbb{R}^2$ open,
$u : \Omega \rightarrow \mathbb{R}$ twice diff

Def**: $u$ is **harmonic** in $\Omega$

$\iff$

$U_{xx}(x,y) + U_{yy}(x,y) = 0$ in $\Omega$.

Laplace's equation!

Theorem: If $f(z) = u(x,y) + iv(x,y)$
is analytic in a domain $D \subseteq \mathbb{C}$,
then $u(x,y)$ and $v(x,y)$ are harmonic
in $D$.

Proof: We'll show in a later lecture
that $u$ & $v$ have partial derivatives of
any order (see sections 54-57 in
the book), for now we will just assume
that the second derivatives exist.

C.-R. eqns: $U_x = V_y$, $U_y = -V_x$

Recall $V_{yy} = V_{yx}\quad \rightarrow\quad U_{xx} = V_{yx}$, $U_{yy} = -V_{xy}$

$\rightarrow\quad U_{xx} + U_{yy} = 0$. 1.
Similarly for \( v \), \( u_{xy} = v_{yy}, \ u_{yx} = -v_{xx} \)

\[ \rightarrow \quad v_{xx} + v_{yy} = 0. \]

**Example:** \( v(x,y) = e^x \sin y \)

\( v_{xx} = e^x \sin y, \ v_{yy} = -e^x \sin y \)

\[ \rightarrow \quad v_{xx} + v_{yy} = 0 \quad \text{\( v \) is harmonic of course!} \]

\( e^z \) is entire and \( e^z = e^x \cos y + ie^x \sin y \)

**Defn:** If \( u \) & \( v \) are harmonic in \( D \) then \( v \) is a harmonic conjugate of \( u \)

the C.R. eqns. \( u_x = v_y, \ u_y = -v_x \) hold.

(This definition was made so that we would have the following theorem:)

**Theorem:** \( v \) is a harmonic conjugate of \( u \)

\[ f(z) = u(x,y) + iv(x,y) \text{ is analytic.} \]

**Examples:**

1. \( e^x \sin y \text{ is a harmonic conjugate of } e^x \cos y. \) (See above.)
2. \( u(x,y) = 2xy \) is a harmonic function.

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial y^2} = 0 \\
\text{so } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0
\end{align*}
\]

Find its harmonic conjugates in \( \mathbb{R}^2 \),
that is, find all \( v(x,y) \) such that

\( u_x = v_y \) and \( u_y = -v_x \).

\[\Rightarrow \text{ solve } \begin{align*}
& v_x = -2x, \quad v_y = 2y \\
\implies & v(x,y) = -x^2 + \Phi(y)
\end{align*}\]

\[v_y = \Phi'(y) = 2y \implies \Phi = y^2 + C.
\]

So \( v(x,y) = y^2 - x^2 + C \).

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**Note:**

\[u(x,y) + iv(x,y) = 2xy + i(y^2 - x^2) + iC
\]

\[= -i(x+iy)^2 + iC
\]

\[= -i \bar{z}^2 + iC
\]

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**Remark:**

If \( v \) is a harmonic conjugate of \( u \),
then \( -u \) is a harmonic conjugate of \( v \).

(If \( u + iv \) is analytic then so is \( -i(u+iv) = v + i(-u) \).
Question (for later):
Does every harmonic function in two variables arise as the real part of an analytic function?

→ equivalently:
Does every harmonic function in two variables have a harmonic conjugate?