10. **Complex Integrals & Contours**

1. Complex valued functions of a real variable

   \([a, b] \subseteq \mathbb{R}\) closed interval

   Consider \( w : [a, b] \rightarrow \mathbb{C} \)

   \[ t \mapsto w(t) \]

   Let \( u(t) = \text{Re}\ w(t) \), \( v(t) = \text{Im}\ w(t) \).

   \[ \Rightarrow w(t) = u(t) + i v(t) \]

   **Derivatives:** \( w'(t) = u'(t) + i v'(t) \)

   **Integrals:** \( \int_a^b w(t) \, dt = \int_a^b u(t) \, dt + i \int_a^b v(t) \, dt \)

   **Def.** \( W(t) \) is an antiderivative of \( w(t) \)

   \( \uparrow \)

   \[ W'(t) = w(t) \]

   **Fundamental Theorem of Calculus:**

   \[ \int_a^b w(t) \, dt = W(b) - W(a) \quad \text{for antiderivative of} \ w(t) \]

   **Proof:** Apply real variable F.T.C. to \( u(t) \) and \( v(t) \).
Examples:

1. Take $z_0 = x_0 + iy_0 \in \mathbb{C}$, fixed. Then $\frac{d}{dt}(e^{z_0 t}) = z_0 e^{z_0 t}$.

To see this, write

$$e^{z_0 t} = e^{x_0 t} e^{iy_0 t} = e^{x_0 t} \cos(y_0 t) + ie^{x_0 t} \sin(y_0 t)$$

and apply $\frac{d}{dt}$. \hfill \text{exercise!}

2. $\int_0^1 (3it^2 + 2i) \, dt = \left[ i t^3 + 2it \right]_0^1 = (i + 2i) - (0 + 0) = 3i$.

3. $\int_0^{\frac{\pi}{2}} e^{it} \, dt = \left[ \frac{e^{it}}{i} \right]_0^{\frac{\pi}{2}} = \frac{e^{i\frac{\pi}{2}} - e^0}{i} = \frac{i - 1}{i} = 1 + i$

Useful Inequality:

$$\left| \int_a^b w(t) \, dt \right| \leq \int_a^b |w(t)| \, dt$$

Proof: Apply Triangle Ineq.
to Riemann sums and take limit.

\hspace{1em} cancellation possible \hspace{1em} no cancellation possible
Standard Identities:

\[ \int_a^b w(t) \, dt = \int_a^c w(t) \, dt + \int_c^b w(t) \, dt \]

\[ \int_a^b w(t) \, dt = - \int_b^a w(t) \, dt. \]

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2. **Contours** (Curves)

\( t \in [a,b] \) \quad \gamma(t) = x(t) + iy(t) \\
\( x(t), y(t) \) continuous real functions

**Def**: \( \gamma(t) \) is an **arc** or a **curve**.

\( \gamma(t) \) is a **simple arc** or **simple curve**

\[ \iff \gamma(t) \text{ does not cross itself.} \]

\[ \iff \gamma(t_1) \neq \gamma(t_2) \text{ if } t_1 \neq t_2. \]

\( \gamma(t) \) is a **simple closed curve** or **Jordan curve**

\[ \iff \gamma(t) \text{ is a simple curve &} \]

\[ \gamma(a) = \gamma(b). \]
Smooth arc/smooth curve:

$z'(t)$ is continuous and nowhere zero.

Contour: a piecewise smooth arc.

Closed contour: Contour + $z(a) = z(b)$

Simple closed contour: Closed contour + Simple

No self-intersections.

Examples:

1. $z(t) = \hat{e}^it$
   $0 \leq t \leq 2\pi$

2. $z(t) = \hat{e}^{-it}$
   $0 \leq t \leq 2\pi$
Length of a smooth arc:
\[ z(t) = x(t) + iy(t), \quad a \leq t \leq b \]

Length
\[ L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \int_a^b |z'(t)| \, dt. \]

→ length of a contour defined by adding the lengths of its smooth pieces.

Reversing orientation:

\[ C \quad \text{parametrized by} \quad z(t), \quad a \leq t \leq b \]
\[ -C \quad \text{parametrized by} \quad w(t) = z(-t), \quad -b \leq t \leq -a. \]

Remarks:
1. \( C \) & \(-C\) are considered different contours.
2. \( \text{Length}(C) = \text{Length}(-C) \).