10. Complex Integrals & Contours

① Complex valued functions of a real variable

$[a,b] \subseteq \mathbb{R}$ closed interval

Consider
$$\begin{aligned} w : [a,b] &\longrightarrow \mathbb{C} \\ t &\longmapsto w(t) \end{aligned}$$

Let $u(t) = \operatorname{Re} w(t)$, $v(t) = \operatorname{Im} w(t)$.

$$\rightsquigarrow \quad w(t) = u(t) + i\,v(t).$$

Derivatives: $\quad w'(t) = u'(t) + i\,v'(t)$

Integrals: $\quad \displaystyle\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

Def^{n}: $W(t)$ is an antiderivative of $w(t)$
$$\Updownarrow$$
$$W'(t) = w(t).$$

Fundamental Theorem of Calculus:

$$\int_a^b w(t)\,dt = W(b) - W(a)$$

$W(t)$ an antiderivative of $w(t)$

Proof: Apply real variable F.T.C. to u & v.

Examples:

① Take $z_0 = x_0 + iy_0 \in \mathbb{C}$, fixed.

Then $\frac{d}{dt}\left(e^{z_0 t}\right) = z_0 e^{z_0 t}$.

To see this, write

$$e^{z_0 t} = e^{x_0 t} e^{i y_0 t} = e^{x_0 t}\cos(y_0 t) + i e^{x_0 t}\sin(y_0 t)$$

and apply $\frac{d}{dt}$. ← exercise!

② $$\int_0^1 (3it^2 + 2i)\,dt = \left[\, it^3 + 2it \,\right]_0^1$$
$$= (i + 2i) - (0 + 0)$$
$$= 3i.$$

③ $$\int_0^{\frac{\pi}{2}} e^{it}\,dt = \left[\frac{e^{it}}{i}\right]_0^{\frac{\pi}{2}}$$
$$= \frac{e^{i\frac{\pi}{2}} - e^0}{i} = \frac{i-1}{i}$$
$$= 1 + i$$

Useful Inequality:

$$\boxed{\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt}$$

cancellation possible

no cancellation possible

Proof: Apply Triangle Ineq. to Riemann sums and take limit.

Standard Identities:

$$\int_a^b w(t)\,dt = \int_a^{\boxed{c}} w(t)\,dt + \int_{\boxed{c}}^b w(t)\,dt$$

$$\int_b^a w(t)\,dt = -\int_a^b w(t)\,dt.$$

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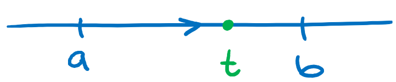
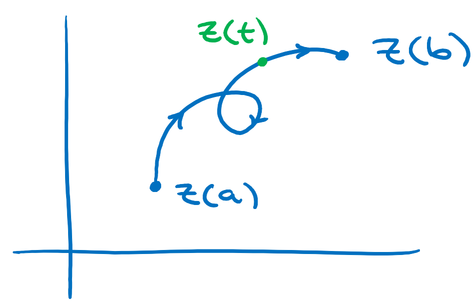
② Contours (Curves)

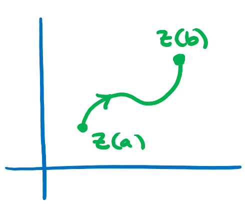
$t \in [a,b]$ $\qquad z(t) = x(t) + i y(t)$

$x(t)$, $y(t)$ continuous real functions

Def^n: $z(t)$ is an arc or a curve.



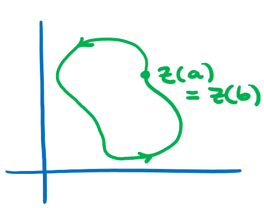




$z(t)$ is a simple arc or simple curve

$\iff$ $z(t)$ does not cross itself.

means $z(t_1) \neq z(t_2)$ if $t_1 \neq t_2$.



$z(t)$ is a simple closed curve or Jordan curve

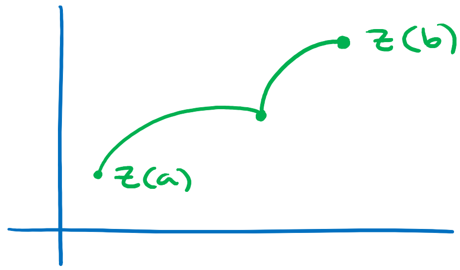
$\iff$ $z(t)$ is a simple curve &

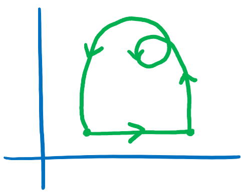
$z(a) = z(b)$.

Smooth arc / smooth curve:

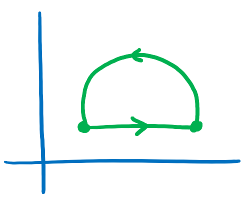
$z'(t)$ is continuous and nowhere zero.

contour: a piecewise smooth arc.





closed contour: contour + $z(a) = z(b)$



simple closed contour: closed contour + simple

no self-intersections

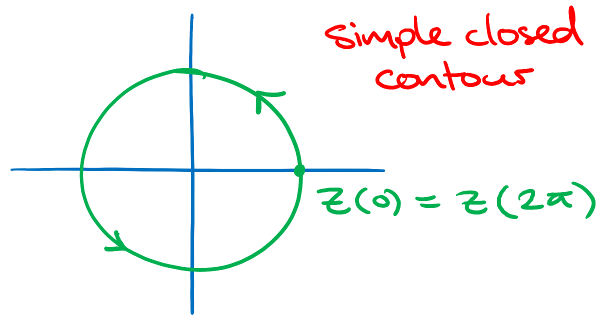
Examples:

Different orientations, so different contours.

① $z(t) = e^{it}$

$0 \le t \le 2\pi$

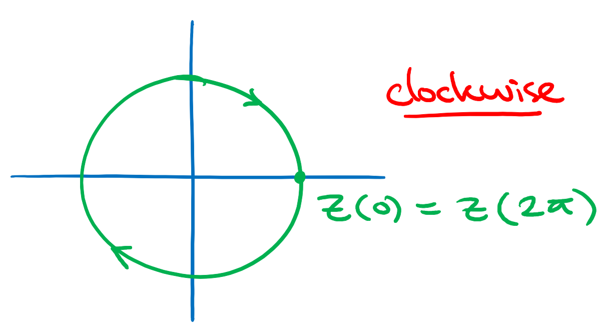
simple closed contour



② $z(t) = e^{-it}$

$0 \le t \le 2\pi$

clockwise



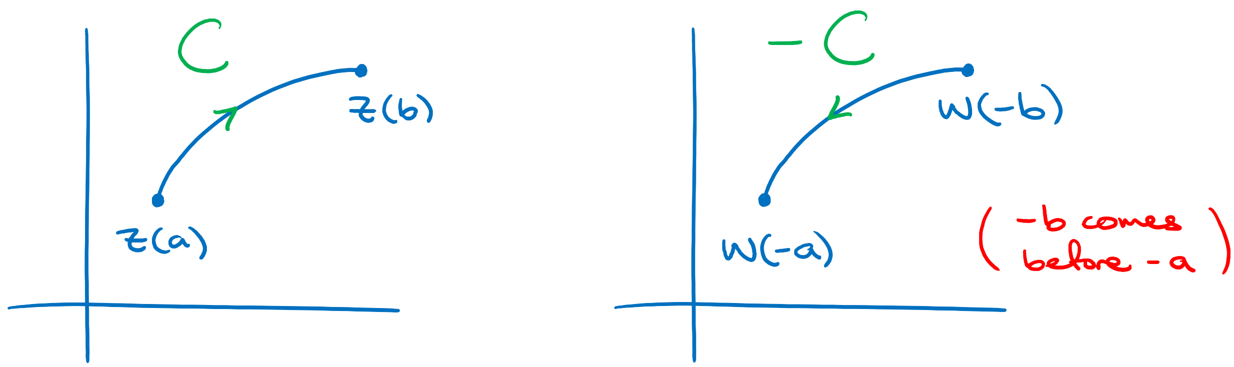
Length of a smooth arc:

$$z(t) = x(t) + i\,y(t), \qquad a \le t \le b$$

length $\quad L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_a^b |z'(t)|\, dt.$

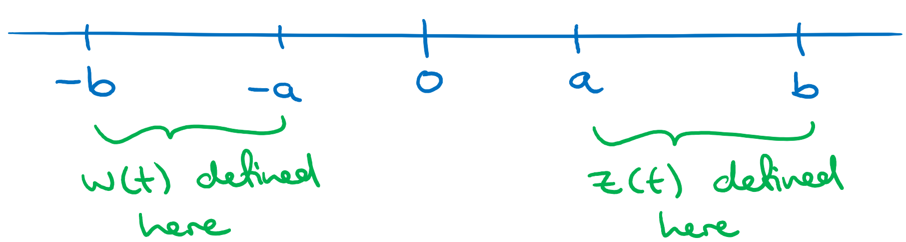
$\rightsquigarrow$ length of a contour defined by adding the lenghts of its smooth pieces.

Reversing orientation:



C : parametrized by $z(t), \quad a \le t \le b$

$-C$: parametrized by $w(t) = z(-t), \; -b \le t \le -a.$



Remarks:

① C & $-C$ are considered different contours.

② Length (C) = Length $(-C)$.