11. Contour Integrals

Contour $C: z(t), \ a \leq t \leq b$

Function $f(z)$ defined on $C$, and (piecewise) continuous on $C$.

Contour Integral:

$$\int_C f(z)\,dz = \int_a^b f(z(t)) \cdot z'(t)\,dt$$

Remarks:

1. If $C = C_1 + C_2$ then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$
2. Following the textbook, we are taking a shortcut with our definition of contour integrals. Usually one defines contour integrals "from first principles" by taking limits of (complex) Riemann sums.

\[ \int_C f(z) \, dz = \lim_{\Delta z_j \to 0} \sum_{j} f(z_j) \Delta z_j \]

If we use the standard definition of contour integrals, then it is a theorem that (if \( C \) is given by \( z(t) \), \( a \leq t \leq b \))

\[ \int_C f(z) \, dz = \int_a^b f(z(t)) z'(t) \, dt. \]

So, our textbook defn is consistent with the standard defn. (agrees)

Exercise: prove this from the textbook defn. up use Chain Rule!

3. If we use the standard definition, then it is clear that \( \int_C f(z) \, dz \) does not depend on the parametrization of \( C \). (\( \int_C f(z) \, dz \) only depends on the orientation of \( C \))
4. Note that we are implicitly using
\[ dz = \frac{dx}{dt} \, dt = z'(t) \, dt \]
to get \[ \int_C f(z) \, dz = \int_a^b f(z(t)) \, z'(t) \, dt. \]

Examples:

1. \( C : z(t) = e^{it}, \quad 0 \leq t \leq \pi. \)
Compute \( \int_C z \, dz. \)

\[ \int_C z \, dz = \int_0^\pi f(z(t)) \, z'(t) \, dt = \int_0^\pi e^{it} \cdot i e^{it} \, dt \]
\[ = \int_0^\pi i e^{2it} \, dt = \left[ \frac{1}{2} e^{2it} \right]_0^\pi \]
\[ = \frac{1}{2} (e^{2i\pi} - e^{2i0}) = \frac{1}{2} (1-1) = 0. \]

2. Same contour. Compute \( \int_C \overline{z} \, dz. \)

\[ \int_C \overline{z} \, dz = \int_0^\pi e^{-it} \cdot i e^{it} \, dt = \int_0^\pi i dt \]
\[ = \left[ it \right]_0^\pi = i\pi. \]
③ \( C: \ z(t) = e^{it}, \ 0 \leq t \leq 2\pi. \)

\[
\int_C \frac{1}{z} \, dz = \int_0^{2\pi} \frac{1}{e^{it}} \cdot i e^{it} \cdot dt = \int_0^{2\pi} i \cdot dt = [it]_0^{2\pi} = 2\pi i.
\]

④ \( C: \ z(t) = e^{-it}, \ 0 \leq t \leq 2\pi \)

\[
\int_C \frac{1}{z} \, dz = \int_0^{2\pi} \frac{1}{e^{-it}} \cdot (-i e^{-it}) \cdot dt = \int_0^{2\pi} (-i) \cdot dt = -2\pi i.
\]

Note that \( C = -C \) (reversed orientation).

\[\Rightarrow \text{In general, for any contour } C, \]

\[
\int_{-C} f(z) \, dz = -\int_C f(z) \, dz.
\]

Proof: \( \text{substitue } t = -t. \)

⑤ \( C: \ z = e^{it}, \ 0 \leq t \leq 4\pi \) (going round the unit circle twice, counter-clockwise)

\[
\int_C \frac{1}{z} \, dz = \int_0^{4\pi} \frac{1}{e^{it}} i e^{it} \, dt = \int_0^{4\pi} i \cdot dt = 4\pi i.
\]
Complex contour integrals and real line integrals:

C: \( z(t) = x(t) + iy(t) \), \( a \leq t \leq b \)

\( f(z) = u(x,y) + iv(x,y) \)

\( dz = dx + idy \)

\[ \Rightarrow \int_C f(z) \, dz = \int_C (u+iv)(dx+idy) \]

\[ = \int_C u \, dx - v \, dy + i \int_C v \, dx + u \, dy \]

Ordinary (real) line integrals!

Recall:

\[ \int_C u \, dx - v \, dy = \int_a^b (ux' - vy') \, dt \]

\[ u(x(t),y(t)) \, x'(t) - v(x(t),y(t)) \, y'(t) \]

\[ \int_C v \, dx + u \, dy = \int_a^b (vx' + uy') \, dt \]

We will make use of this point of view later.
Examples involving branch cuts

Suppose we want to integrate

\[ f(z) = \text{P.V. } z^{1/2} = \exp\left(\frac{1}{2} \log z\right) \]

over the contour \( C: z(t) = e^{it}, 0 \leq t \leq 2\pi. \)

Does the branch cut pose a problem?

No problem!

Option 1: Break integral into 2 parts:

\[ 0 \leq t \leq \pi \quad \text{and} \quad \pi < t \leq 2\pi. \]

Option 2: Since contour integrals do not depend on the parametrization (only the orientation) we could just write \( C \) as \( z(t) = e^{it}, -\pi \leq t \leq \pi. \)

\[ f(z(t)) = \exp\left(\frac{1}{2} \log e^{it}\right) = \exp\left(\frac{1}{2}it\right) \]

for \( -\pi < t < \pi. \)

We do not care about the value at \( t = -\pi \) as it will not affect the integral.
\[ \int_C f(z) \, dz = \int_{-\pi}^{\pi} e^{it} \cdot i e^{it} \, dt = \int_{-\pi}^{\pi} i e^{2it} \, dt \]
\[ = \left[ -\frac{1}{2} e^{2it} \right]_{-\pi}^{\pi} = \frac{2}{3} (e^{2i\pi} - e^{-2i\pi}) = \frac{2}{3} (-1 - i) = -\frac{4i}{3} . \]

**Upper bounds for moduli of contour integrals**

**Another very useful inequality:**

If \( f(z) \) is a contour \( C \), if \( |f(z)| \leq M \) \( \forall z \in C \) and \( \text{Length}(C) = L \) then

\[ |\int_C f(z) \, dz| \leq ML . \]

**Proof:**

\[ |\int_C f(z) \, dz| = \left| \int_a^b f(z(t)) z'(t) \, dt \right| \]
\[ \leq \int_a^b |f(z(t))||z'(t)| \, dt \]
\[ \leq \int_a^b M |z'(t)| \, dt = M \int_a^b |z'(t)| \, dt = ML. \]

\( \square \)

**Examples:**

1. \( C: z(t) = 3e^{it}, \ 0 \leq t \leq \frac{\pi}{2} . \)

\( \Rightarrow \) \( \text{Length}(C) = 3 \times \frac{\pi}{2} = \frac{3\pi}{2} . \)
Find a bound for $|\int_{C_c} (\bar{z}^2 + i) \, dz|$. 

For $z \in \mathbb{C}$, $|\bar{z}^2| = |\bar{z}|^2 = |z|^2 = 3^2 = 9$, 

$\Rightarrow |\bar{z}^2 + i| = |\bar{z}^2| + |i| = 9 + 1 = 10$. 

$\Rightarrow |\int_{C_c} (\bar{z}^2 + i) \, dz| \leq \frac{30\pi}{2} = 15\pi$. 

2. $C_R: z(t) = Re^{it}, \ 0 \leq t \leq \pi$, 

where $R > 1$. 

Claim: $\lim_{R \to \infty} \int_{C_R} \frac{z-2}{z^4+1} \, dz = 0$. 

Proof: Let $I_R = \int_{C_R} \frac{z-2}{z^4+1} \, dz$. 

Note that $\text{Length}(C_R) = \pi R$. 

Recall: $|z_1 + z_2| \leq |z_1| + |z_2|$, 

$|z_1 - z_2| \geq |z_1| - |z_2|$. 

For $z \in \mathbb{C}$, $|z-2| \leq |z| + 2 = R + 2$, 

$|z^4 + 1| \geq |z^4 - 1| = R^4 - 1$. 

$\Rightarrow \left| \frac{z-2}{z^4+1} \right| \leq \frac{R+2}{R^4-1} \text{ for } z \in \mathbb{C}$. 

$\Rightarrow |I_R| \leq \frac{R+2}{R^4-1} \pi R = \pi \frac{R^2 + 2R}{R^4-1}$. 

8.
We get

\[ |I_R| \leq \pi \frac{R^2 + 2R}{R^4 - 1} = \pi \frac{\sqrt{R^2 + 3R^2}}{1 - \frac{1}{R^4}} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty. \]

Multiply numerator and denominator by \( \frac{1}{R^4} \)

Hence \( \lim_{R \rightarrow \infty} IR = 0. \)